A crash-course on equilibration and thermalization in quantum many-body systems

With a focus on rigorous results

Henrik Wilming [ETH Zürich]

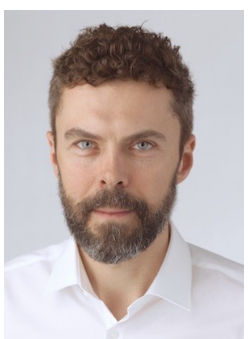
Quantum Thermodynamics for Young Scientists, Bad Honnef, February 4th 2020

A crash-course on equilibration and thermalization in quantum many-body systems

With a focus on rigorous results

DISCLAIMER: I will only give a small peak into the field. What comes will be **incomplete and biased**.
In particular, I will leave out **typicality approaches** to equilibration and thermalization (see, e.g., works by **Reimann, Gemmer** and **Steinigeweg** groups).

Other people at the conference that know more about specific aspects of the problem! If you're interested in the topic, for example, also talk to:



Markus Müller

Luis Pedro García-Pintos

... and the speakers of the Wednesday morning session!

In (quantum) statistical mechanics, we ascribe a statistical ensemble to the state of a system obtain predictions. For example, Gibbs-state:

$$\rho(t) = \omega_{\beta,H} := \frac{\mathrm{e}^{-\beta H}}{Z_\beta}$$

Depending on situation, different justifications can be given, e.g.:

- Complete passivity
  Essentially a thermodynamics argument.
  Don't we want to derive thermo from quantum theory?
- Jaynes' Maximum Entropy principle
  Lack of knowledge, seems "subjective".
- Typicality
  (almost all quantum states in an energy-shell resemble a Gibbs state for physically relevant observables)
  But why aren't the physically relevant initial states in the set of measure zero for which it doesn't apply?

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Topic today: Can we maybe **derive** the **dynamical emergence** of statistical ensembles in quantum mechanics from **reasonable assumptions**?

Need to be able to show three things:

1) System reaches a stationary state in the first place. (**Equilibration**)

2) Stationary state well described by thermal ensemble for physically relevant observables. (**Thermalization**)

3) Equilibration happens in a **reasonable time.**

Some connections to other fields

Quantum gravity (holographic duality)
(Eternal) black holes described by thermal state in dual CFT.
How do thermal states arise?
How does information get scrambled?

Condensed matter physics
Understanding out of equilibrium dynamics.
Understanding possible phases of matter.

Quantum thermodynamics
Understanding strong coupling corrections to thermodynamic bounds.
Understanding when thermal states are applicable.

Quantum simulation
Simulating dynamics of complex Quantum systems in experiments.

1. Equilibration

Basic questions:

- How does it happen (if it happens)?
- Does it actually happen (even in infinite time)?
- How long does it take?

Big open problem!

Will only briefly touch it at the end.

Talk to Luis Pedro García-Pintos!

Reviews:
- Nature Physics 11, 124 (2015) Eisert, Friesdorf, Gogolin
- Advances in Physics 65, 3 (2016) D'Alessio, Kafri, Polkovnikov, Rigol
- Rep. Prog. Phys. 79, 056001 (2016) Gogolin, Eisert

For pedagogical explanation, see for example:
- H.W., T. R. de Oliveira, A. J. Short, J. Eisert, in "Thermodynamics in the quantum regime", F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso (eds.) (Springer, Berlin, 2018)
- H.W., M. Goihl, C. Krumnow, J. Eisert, arXiv:1704.06291
- T. R. de Oliveira, C. Charalambous, D. Jonathan, M. Lewenstein, A. Riera, New Journal of Physics 20, 033032 (2018)

See, also e.g., work by the Short group (Bristol), Reimann group (Bielefeld), Gemmer group (Osnabrück), Eisert group (Berlin), Abanin, Calabrese, Cardy, Caux, Essler, Farelly, Gogolin, Kastner, Masanes, Müller, Osborne, Popescu, Winter and many more

Consider a **finite** quantum system evolving under a Hamiltonian H:

$$|\Psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\Psi(0)\rangle$$

Also consider a set of "physically relevant" observables $\mathcal{A}$.

Is it true that (after sufficiently long time) $\langle\Psi(t)|A|\Psi(t)\rangle \approx \mathrm{Tr}[A\overline{\rho}]$ **for all observables in** $\mathcal{A}$ **?**

If system equilibrates, then the stationary state is necessarily well approximated by the time-averaged state:

$$\overline{\rho} = \overline{|\Psi(t)\rangle\langle\Psi(t)|}^{\infty} := \lim_{T\to\infty}\frac{1}{T}\int_0^T |\Psi(t)\rangle\langle\Psi(t)|$$

$$= \sum_i p_{E,i}|E_i\rangle\langle E_i|$$

For pure initial states

Probability distribution of energy

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In general: **No!**

Counter-examples:

- Single spin in a magnetic field that keeps rotating.
- Harmonic oscillator in a coherent state, oscillating back and forth.

Recurrence time: Unitary dynamics is reversible and state-space compact. Therefore, for any $\epsilon > 0$ there exists a time T such that

$$\|\rho(T) - \rho(0)\|_1 \leq \epsilon$$

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Counter-examples:

- Single spin in a magnetic field that keeps rotating.
- Harmonic oscillator in a coherent state, oscillating back and forth.

Not the kind of systems, we usually observe to equilibrate: What about large, interacting systems?

Recurrence time in finite-dimensional systems:

Unitary dynamics is reversible and state-space compact.
Therefore, for any $\epsilon > 0$ there exists a time T such that $\|\rho(T) - \rho(0)\|_1 \leq \epsilon$

But T ~ exp(exp(N)), hence irrelevant (just like in classical mechanics!).
Sufficient if system is close to a stationary value for most times.

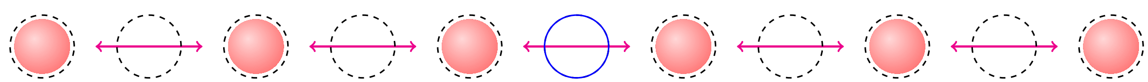
Basics

Consider a quantum system evolving under a Hamiltonian H:

$$|\Psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\Psi(0)\rangle$$

Also consider a set of "physically relevant" observables $\mathcal{A}$.

Do the expectation values of these observables become stationary?



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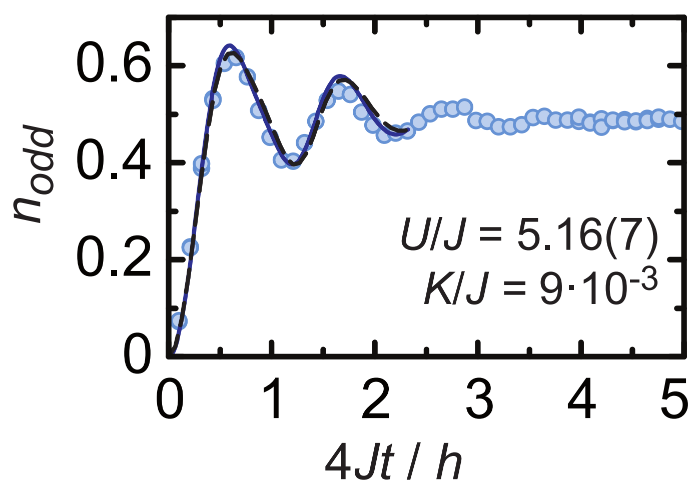

S. Trotzky, Y-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert and I. Bloch, Nature Physics 8, 325–330 (2012)

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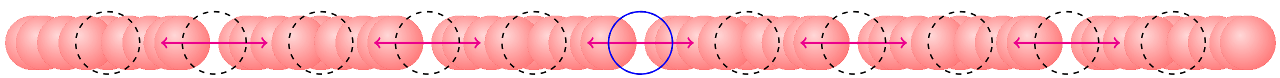

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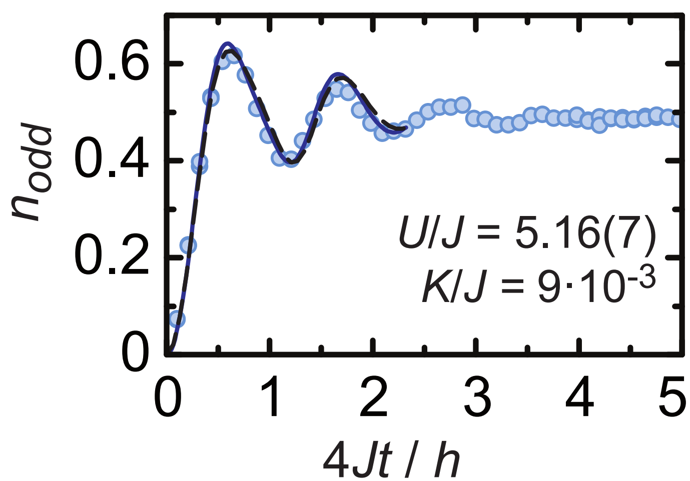

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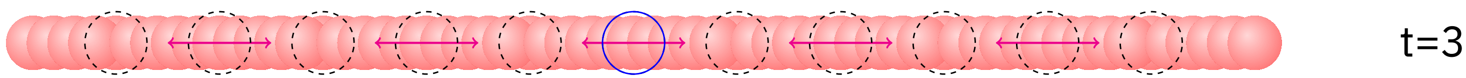

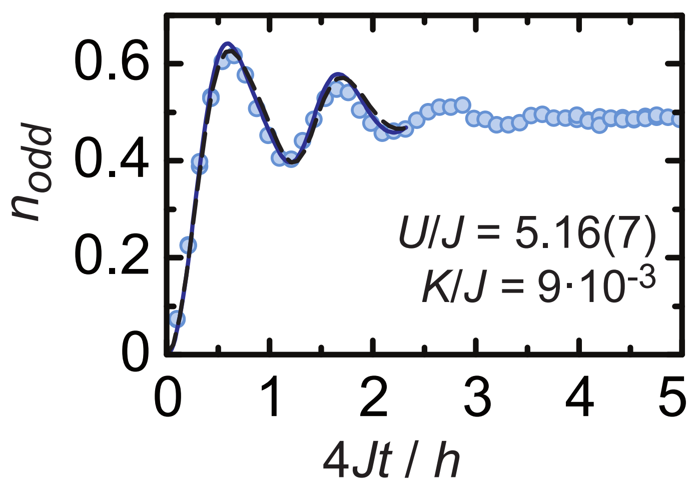

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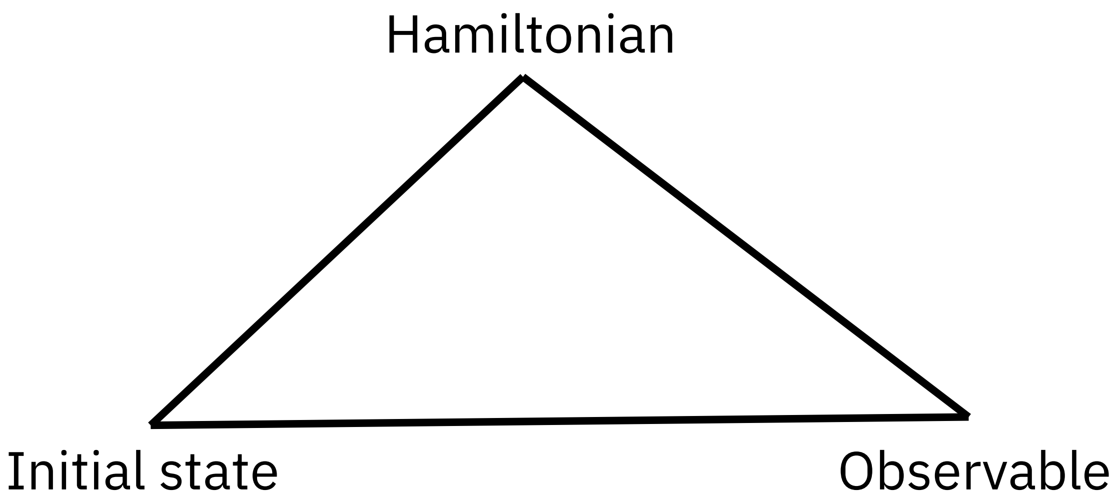
Do the expectation values of these observables become stationary?

Depends on Hamiltonian, initial state and set of observables!



Equilibration: How does it happen? Intuition: Dephasing.

$$|\Psi(t)\rangle = \sum_i \mathrm{e}^{-\mathrm{i}E_i t} c_i |E_i\rangle$$

Can decompose time-dependent expectation value of an observable in energy-eigenbasis:

$$\langle\Psi(t)|A|\Psi(t)\rangle - \mathrm{Tr}[A\overline{\rho}] = \sum_{j\neq k} \mathrm{e}^{-\mathrm{i}(E_j - E_k)t} v_{j,k}$$

$$= \sum_{G_\alpha \neq 0} \mathrm{e}^{-\mathrm{i}G_\alpha t} z_\alpha,$$

$$v_{j,k} = c_j c_j^* A_{j,k}$$

$$z_\alpha = \sum_{E_j - E_k = G_\alpha} v_{j,k}$$

Collection of points in the complex plane moving rotating with angular velocities G_α ("energy gaps") on circles of radius $|z_\alpha|$.

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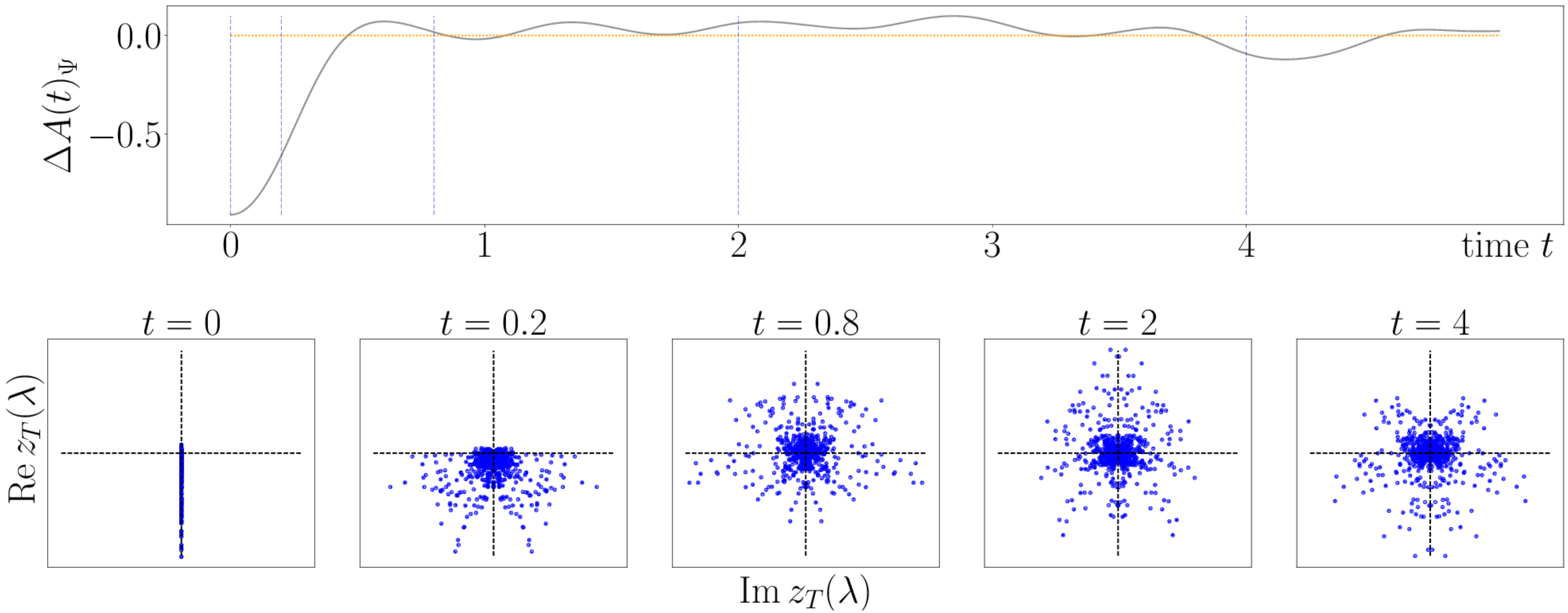


Intuitively, for good equilibration, want:

1) Many points: **large Hilbert-space dimension**
2) Moving with different velocities (**a variety of energy-gaps**)
3) None of them should be on a circle with very large radius (**weight evenly distributed**)

It's a very special situation not to be equilibrated already!

Equilibration: Does it happen in infinite time? - A theorem

Theorem: For any finite-dimensional system, any observable and any initial state:

$$\overline{(\mathrm{Tr}[A\rho(t)] - \mathrm{Tr}[A\overline{\rho}])^2}^{\infty} \leq \|A\|^2 D_G \mathrm{e}^{-S_2(\vec{p}_E)}$$

Here:

- Maximum **degeneracy of energy gaps**

$$D_G = \max_{j,k} |\{(l,m) \; : \; E_l - E_m = E_j - E_k\}|$$

- Probability distribution of energy:

$$p_{E,i} = \mathrm{Tr}[P_i \rho(0)], \quad H = \sum_i E_i P_i$$

- 2nd Rényi entropy:

$$S_2(\vec{p}) = -\log\left(\sum_i p_i^2\right) \leq S(\vec{p}) = -\sum_i p_i \log(p_i)$$

- Infinite time-average:

$$\overline{f(t)}^{\infty} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t) \mathrm{d}\, t$$

P. Reimann, Phys. Rev. Lett. 101, 190403 (2008), arXiv:0810.3092 .
A. J. Short and T. C. Farrelly, New J. Phys. 14, 013063 (2012), arXiv:1110.5759 .

Good equilibration

if

Few degenerate gaps (“generically true in interacting systems”)

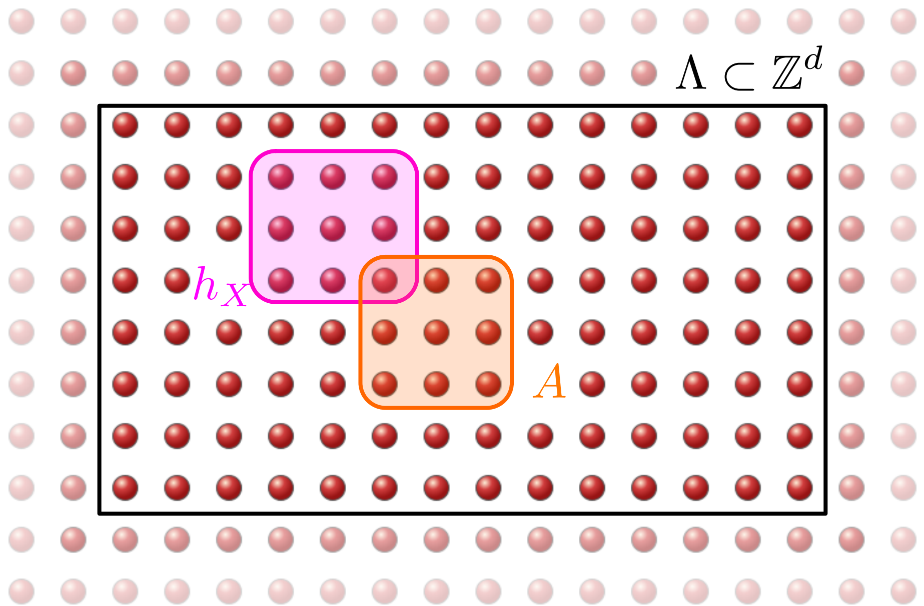
+

Energy distribution spread evenly over many different energy-levels

To show this, need additional assumptions.

Equilibration: When is $S_2(\vec{p}_E)$ big?

From now on: Consider many-body systems modelled by interacting spins on a lattice



Hilbert-space:

$$\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathcal{H}_x, \qquad |\Lambda| = N$$

Hamiltonian

$$H_\Lambda = \sum_{X\subseteq\Lambda} h_X$$

Interactions **k-local:**

$$h_X = 0 \quad \text{if} \quad \mathrm{diam}(X) \geq k.$$

Physically relevant observables:

i) local observables, ii) extensive observables, for example: $S^z = \sum_{x\in\Lambda} \sigma_x^z$

Physically relevant initial states: States with **finite correlation length:**

$$\mathrm{Cov}(A,B)_\rho := |\mathrm{Tr}[\rho AB] - \mathrm{Tr}[\rho A]\mathrm{Tr}[\rho B]| \leq \|A\|\|B\|\mathrm{e}^{-d(A,B)/\xi}$$

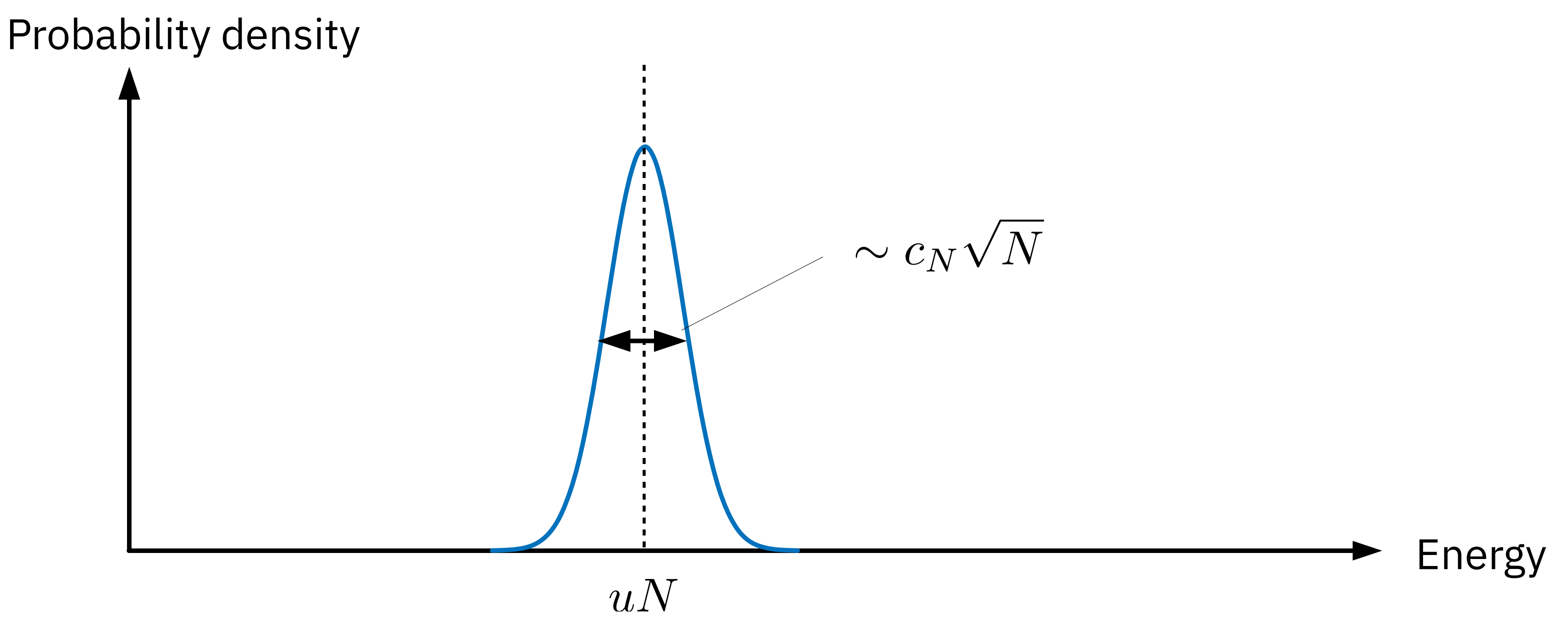
Examples: Product states, Gibbs states above critical temperature,
Ground states of gapped Hamiltonians, generic matrix product states

Equilibration: When is $S_2(\vec{p}_E)$ big? - A theorem

If state has finite correlation length, energy disitribution of a local Hamiltonian is **roughly Gaussian**.

$$\longrightarrow \qquad c_N^2 := \mathrm{Cov}(H,H)_\rho / N \leq O(1)$$

Typically, $c_N \geq c > 0$ (unless eigenstate).



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Theorem: If a state has finite correlation length and the Hamiltonian is k-local, then

$$S_2(\vec{p}_E) \geq \log(c_N^3\sqrt{N}) - \log(C\log^{2d}(N)).$$

Hence:

$$\overline{(\mathrm{Tr}[A\rho(t)] - \mathrm{Tr}[A\overline{\rho}])^2}^{\infty} \leq \|A\|^2 D_G \frac{C\log^{2d}(N)}{c_N^3\sqrt{N}} \quad \xrightarrow{N\to\infty} 0$$

T. Farrelly, F. G. S. L. Brandao, M. Cramer, Phys. Rev. Lett. 118, 140601 (2017)

System has finite correlation length and is not in eigenstate of Hamiltonian.

$$\downarrow$$

Equilibration (in infinite time) in case of few degenerate energy gaps.

Application (Stability of non-critical thermal states): Suppose system is thermal initially with finite correlation length. Any local perturbation can change the entropy only by an amount O(1). Then system will equilibrate again. In fact will thermalize "on average" (see paper).

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Remark: The Theorem only gives a **vanishing entropy density** in thermodynamic limit:

$$\lim_{N\to\infty} \frac{1}{N} S_2(\vec{p}_E) = 0$$

2. Thermalization

Basic question:
Assuming a system equilibrates, when does it also thermalize?

Reviews:
- Nature Physics 11, 124 (2015) Eisert, Friesdorf, Gogolin
- Advances in Physics 65, 3 (2016) D'Alessio, Kafri, Polkovnikov, Rigol
- Rep. Prog. Phys. 79, 056001 (2016) Gogolin, Eisert

Some key papers:
- M. Srednicki, Phys. Rev. E 50 (1994)
- M. Rigol, V. Dunjko, and M. Olshanii, Nature 452 (2008)
- S. Popescu, A. J. Short, and A. Winter, Nature Phys. 2 (2006)
- M. P. Mueller, E. Adlam, L. Masanes, N. Wiebe, Communications in Mathematical Physics 340 2 (2015)
- T. Farrelly, F. G. S. L. Brandao, M. Cramer, Phys. Rev. Lett. 118, 140601 (2017)

Thermalization: Basics

Assume many-body system equilibrates from some initial state for local observables.
We say it **thermalizes**, if also (for local observables)

$$\mathrm{Tr}[A\overline{\rho}] \approx \mathrm{Tr}[A\frac{\mathrm{e}^{-\beta H_\Lambda}}{Z_\beta}]$$

Typically require that the error **decreases with increasing system-size**.

Caution, in strongly coupled systems, reduced state of global thermal state:

$$\mathrm{Tr}_{A^c}[\overline{\rho}] \approx \mathrm{Tr}_{A^c}[\frac{\mathrm{e}^{-\beta H_\Lambda}}{Z_\beta}] \neq \frac{\mathrm{e}^{-\beta H_A}}{Z_{A,\beta}}$$

Assume man... system equilibrates from some initial state for local observables.

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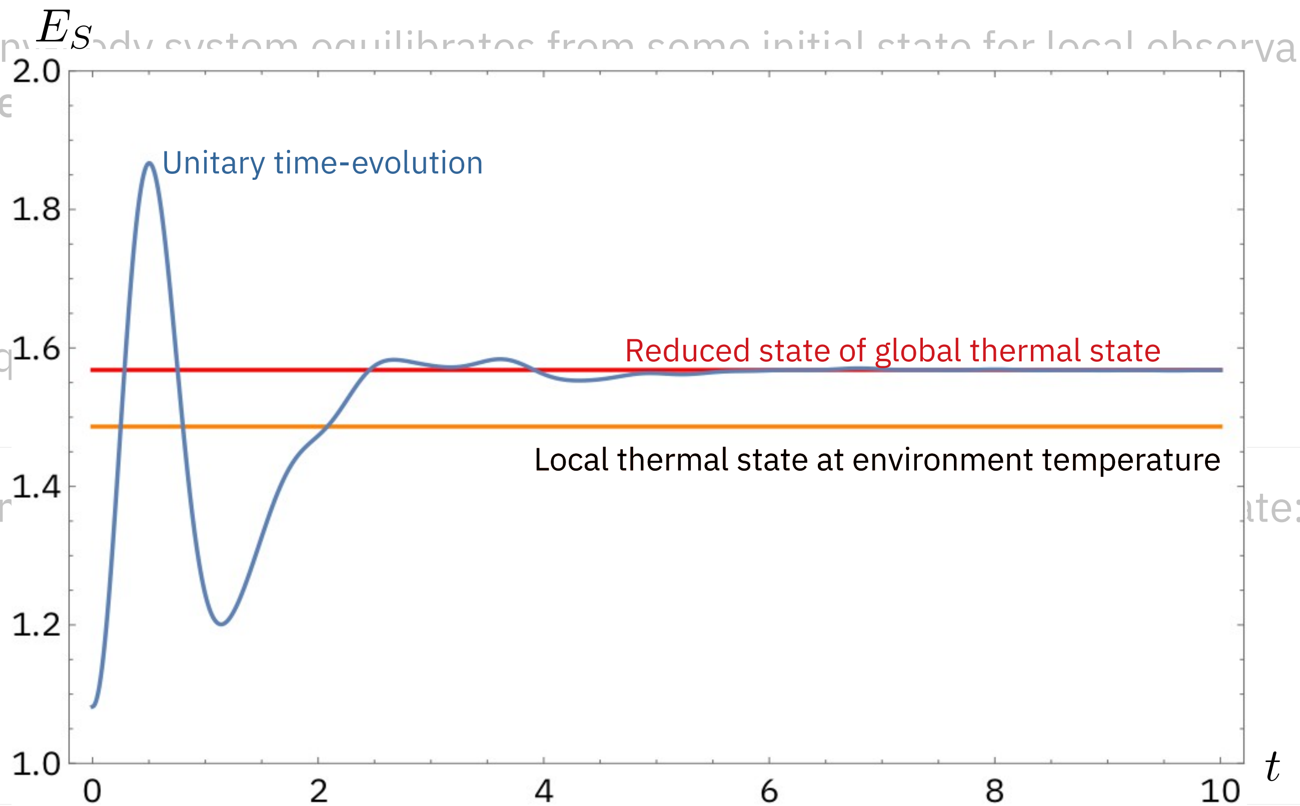
Typically req...

Caution ... ate:



Average energy of central oscillator over time in Caldeira-Leggett model with ohmic spectral density

M. Perarnau-Llobet, H. Wilming, A. Riera, R. Gallego, and J. Eisert, *Phys. Rev. Lett. 120, 120602* (2018)

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Reminder (Gibbs' principle): Given a state and Hamiltonian of a finite system. Let

$u = \mathrm{Tr}[H_\Lambda \rho]/N$: Expected **energy density** for a given state and Hamiltonian

$s = S(\rho)/N$: von Neumann **entropy density** of state

The Gibbs-state at inverse temperature 1/T is the **unique state** that minimizes the **free energy density**

$$f_T = u - Ts$$

Expected energy unchanged by evolution, but entropy of equilibrium state can be much larger than initial value. **Does thermalization follow as long as entropy density of equilibrium state large enough?**

Thermalization: An informal theorem

Theorem (highly informal): Consider translationally invariant systems of increasing size N with non-degenerate spectra. Further consider a sequence of initial states with the following properties:

1) The energy density of the states converges and there is a unique thermal state at the corresponding temperature in the thermodynamic limit (i.e., above phase transition).
2) The entropy density of the time-averaged states converges to the thermal one.

Then local observables in equilibrium are described to arbitrary accuracy (with increasing N) by the thermal Gibbs state in the thermodynamic limit.

M. P. Mueller, E. Adlam, L. Masanes, N. Wiebe,
Communications in Mathematical Physics 340 2 (2015)

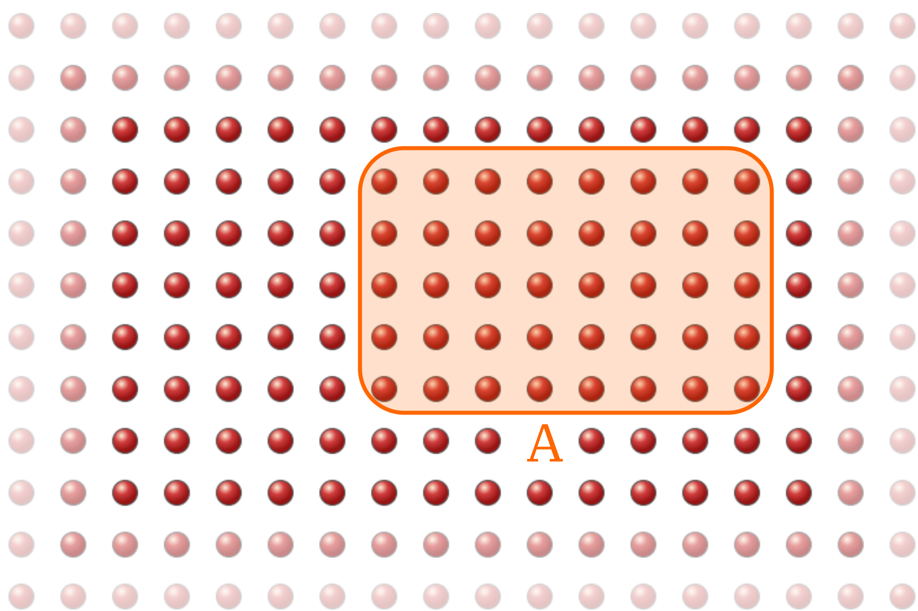
T. Farrelly, F. G. S. L. Brandao, M. Cramer,
Phys. Rev. Lett. 118, 140601 (2017)

Roughly: Under weak assumptions, if the energy distribution of initial state has large enough entropy, system will locally thermalize.

Need to be able to argue that the energy distribution for reasonable initial states has maximum entropy density!

A new ingredient: Entanglement in energy eigenstates.

Interlude: Entanglement in many-body systems



Consider an energy eigenstate $|E_i\rangle$ of a many-body system.

$$\rho(A)_i = \mathrm{Tr}_{\Lambda \setminus A}[|E_i\rangle\langle E_i|]$$

What's the entropy of the reduced density matrix on a large region A?

A **random pure state** fulfills:

$$S(\rho(A)) \sim |A| = \mathrm{Volume}(A)$$

General believe in "ergodic systems":

At **very low** energies (u=0 in thermodynamic limit): $S(\rho(A)_i) \sim |\partial A| = \mathrm{SurfaceArea}(A)$

"Area Law"

At **high** energies (u>0 in thermodynamic limit): $S(\rho(A)_i) \sim |A| = \mathrm{Volume}(A)$

(In this regime, properties of many-body systems are often well-described by random matrices.)

"Volume Law"

Entanglement-ergodicity

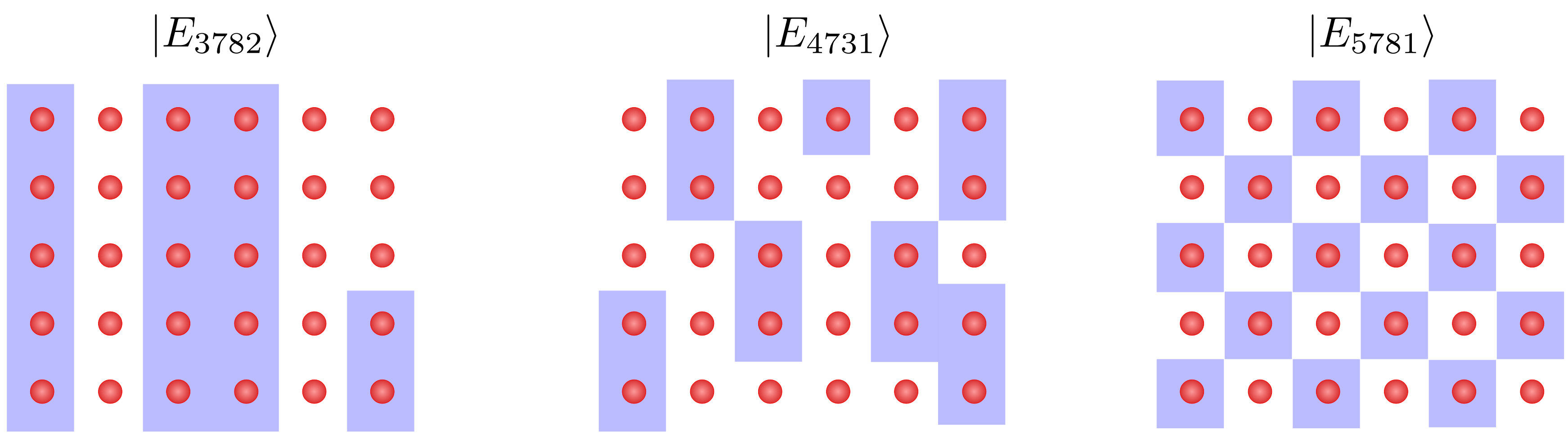
Informal definition: Entanglement-ergodicity

Call a sequence of spin-lattice systems of increasing system size N **entanglement-ergodic** if for **every energy eigenstate** |E> with **positive energy density** the 2nd Renyi **entanglement entropy** of **some finite fraction A(E)** of the lattice is **extensive:**

$$S_2(\rho(A(E))) \geq g(E/N)N$$

g is a sufficiently regular function of the energy density (e.g., Lipschitz)

H.W., M. Goihl, I. Roth, J. Eisert, Phys. Rev. Lett. 123, 200604 (2019)



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H.W., M. Goihl, I. Roth, J. Eisert, Phys. Rev. Lett. 123, 200604 (2019)

- Intuitively: every energy eigenstate with positive energy density fulfills a **weak volume law.**
- Very forgiving: considered subsystem need **not** be **connected**. **Even generic matrix product states are expected fulfill such a weak volume law!** (Important: MBL systems equilibrate, but don't thermalize)
- Volume law for **von Neumann** entropy **not enough** (counter-example).
- Plausibly implied by **Eigenstate Thermalization Hypothesis (ETH)** (open problem, comes later)
- Equivalent definition for any Renyi entropy with $\alpha > 1$.

Entanglement-ergodicity: Implications

Informal Theorem

In an **entanglement-ergodic system every initial pure product state** with **extensive energy** has extensive (Renyi) entropy in the energy distribution (for large enough N):

$$S(\vec{p}_E) \geq S_2(\vec{p}_E) \geq kN,$$

where k>0 is a constant. Hence, **entropy density is positive in equilibrium**.

Corollary

In an entanglement-ergodic system with few degenerate energy-gaps, **every pure initial product state with extensive energy equilibrates to exponential precision:** For any observable A,

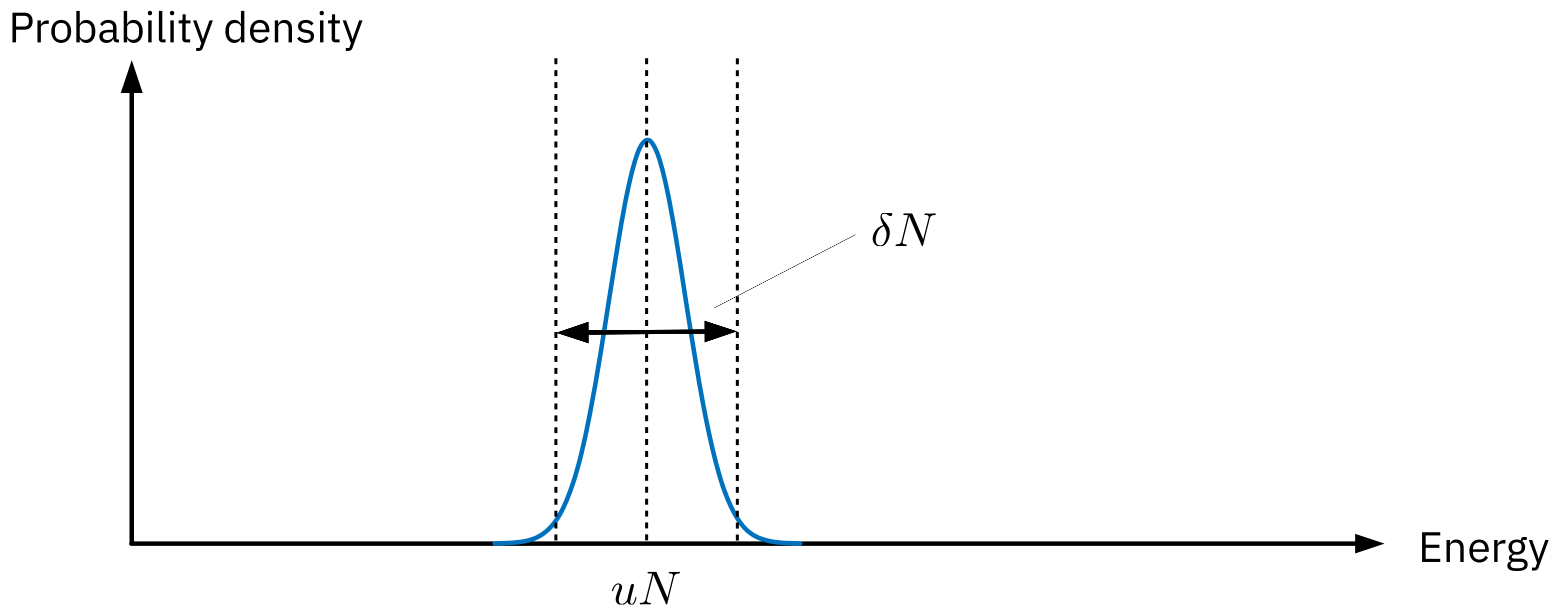
$$\overline{(\langle A(t)\rangle - \mathrm{Tr}[A\overline{\rho}])^2}^{\infty} \leq \|A\|^2 D_G \mathrm{e}^{-kN}.$$

Open problem: Prove that k is large enough to obtain thermalization under suitable assumptions.

H.W., M. Goihl, I. Roth, J. Eisert, Phys. Rev. Lett. 123, 200604 (2019)

Entanglement-ergodicity: Proof sketch

$$S_2(\vec{p}_E) \geq kN \quad \Leftrightarrow \quad \max_i p_{E,i} \leq \mathrm{e}^{-kN}$$



- Outside window: Energy distribution ~ Gaussian, hence total probability outside of window exponentially small
- Inside window: Use extensive entanglement entropy to show that overlap of energy eigenstates with any product state is exponentially small.

H.W., M. Goihl, I. Roth, J. Eisert, Phys. Rev. Lett. 123, 200604 (2019)

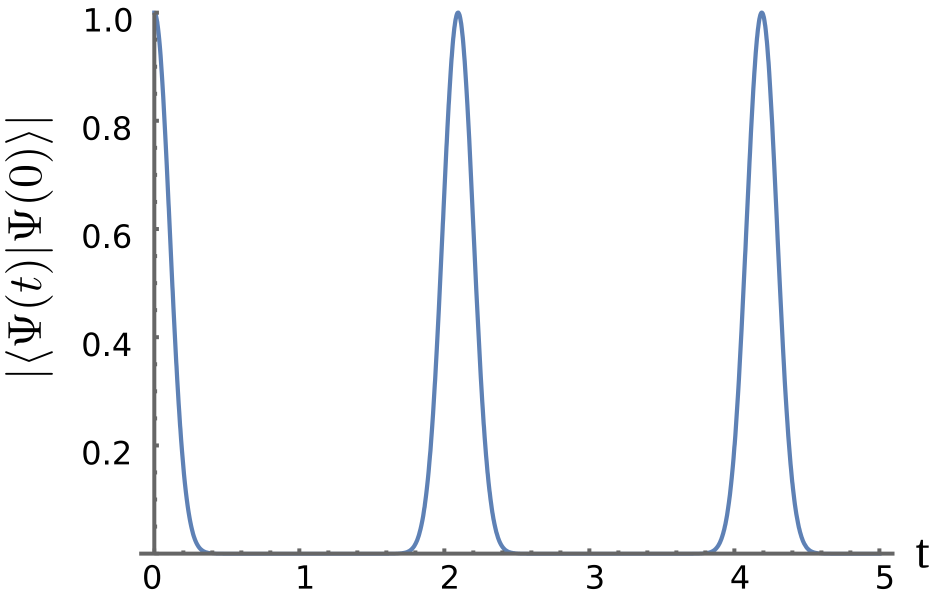
Are the assumptions necessary? - Quantum many-body scars

Two crucial assumptions: 1) Few degenerate energy gaps

2) Entanglement in energy-eigenstates high
(to get high entropy of energy distribution)

In the last two years, very interesting development. There exist k-local, interacting, “non-integrable” Hamiltonians with **perfect revival of initial product state:**

$$|\Psi(0)\rangle = |\phi\rangle^{\otimes N} \qquad \text{and} \qquad |\langle\Psi(\tau)|\Psi(0)\rangle| = 1, \quad \tau = \text{const.}$$



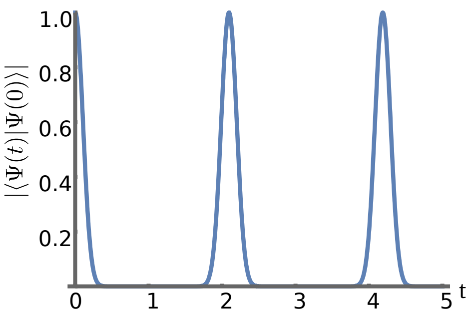
See, for example, Choi et al. Phys. Rev. Lett. 122, 220603 (2019)

Are the assumptions necessary? - Quantum many-body scars

Two crucial assumptions: 1) Few degenerate energy gaps

2) Entanglement in energy-eigenstates high
(to get high entropy of energy distribution)

In the last two years, very interesting development. There exist k-local, interacting, “non-integrable” Hamiltonians with **perfect revival of initial product state:**



$$|\Psi(0)\rangle = |\phi\rangle^{\otimes N} \qquad \text{and} \qquad |\langle\Psi(\tau)|\Psi(0)\rangle| = 1, \quad \tau = \text{const.}$$

What goes “wrong” in these models? It is found in the particular models that there exist O(N) eigenstates $|E_i\rangle$ such that:

1) $$E_{i+1} - E_i = \Delta = \text{const.}$$

N degenerate energy gaps, related to an “emergent SU(2) symmetry”

2) The energy eigenstates have very little entanglement, even though energy density is positive!

$$S(\rho(A)_i) \sim \log(|A|)$$

“Perfect quantum many-body scars”

See, for example, Choi et al. Phys. Rev. Lett. 122, 220603 (2019)

Quantum many-body scars: A theorem

Informal Theorem: “Perfect revivals imply quantum many-body scars”

Consider any k-local Hamiltonian and assume there exists an initial state such that

$$|\Psi(0)\rangle = |\phi\rangle^{\otimes N} \qquad \text{and} \qquad |\langle\Psi(\tau)|\Psi(0)\rangle| = 1, \quad \tau = \mathrm{poly}(N)$$

Then there exist at least $\tilde{O}(\tau\sqrt{N})$ many energy **eigenstates with energies in an equally-spaced set of energies** and **each of them fulfills**:

$$S_\alpha(A) \leq O(\log(|A|)), \quad \forall\alpha > 1, \quad \forall A \subseteq \Lambda$$

A. M. Alhambra, H.W., arXiv: 1911.05637

- Assumption on state can be relaxed to “area law + finite correlation length”
- Energy-spacing ~ $1/\tau$.
- In case of imperfect revivals: Scar-states which are approximate eigenstates.

The eigenstate thermalization hypothesis

So far: wanted to show thermalization using Gibbs' principle

Imagine instead that **every energy eigenstate at high energy locally** looks **thermal:**

$$\langle E_i|A|E_i\rangle \approx \mathrm{Tr}\left[A\frac{\mathrm{e}^{-\beta_i H}}{Z_{\beta_i}}\right] \qquad \text{with } \beta_i \text{ s.t.} \qquad \langle E_i|H|E_i\rangle = \mathrm{Tr}\left[H\frac{\mathrm{e}^{-\beta_i H}}{Z_{\beta_i}}\right]$$

"Eigenstate Thermalization Hypothesis (ETH)"
(Different formulations exist.)

If:
1) System equilibrates,
2) Probability distribution of energy **peaked at energy density u**
(for example, state with finite correlation length),

Then:

$$\mathrm{Tr}[A\overline{\rho}] \approx \mathrm{Tr}\left[A\frac{\mathrm{e}^{-\beta H}}{Z_{\beta}}\right] \qquad \text{with } \beta \text{ s.t.} \qquad uN = \mathrm{Tr}\left[H\frac{\mathrm{e}^{-\beta H}}{Z_{\beta}}\right]$$

M. Srednicki, Phys. Rev. E 50 (1994)
M. Rigol, V. Dunjko, and M. Olshanii, Nature 452 (2008)
D'Alessio, Kafri, Polkovnikov, Rigol, Advances in Physics 65, 3 (2016)

The eigenstate thermalization hypothesis: pros and cons.

Pro	Contra
Amazingly elegant explanation of Thermalization!	Need a good explanation why ETH should hold.
Fair amount of numerical evidence in favor of ETH for “non-integrable”, interacting systems.	No generally accepted definition of “non-integrable” systems in quantum theory. Also: “Counter-examples” known (many-body localization, many-body scars)
Has led to many interesting insights and observations.	Many different variations of ETH available. Not precisely formulated for which observables it is supposed to hold. Makes it difficult to prove or disprove.

Interesting open problem: Prove that a suitable formulation of ETH implies entanglement-ergodicity. If true, ETH would already imply equilibration.

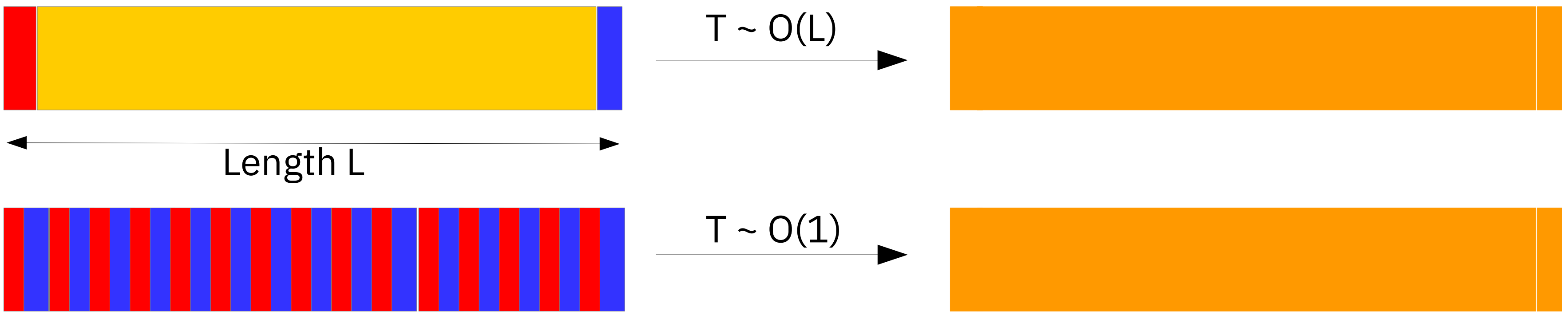
3. How long does equilibration take?

Short, pedagogical review on known results (includes Refs. below):
H.W., T. R. de Oliveira, A. J. Short, J. Eisert, in "Thermodynamics in the quantum regime", (Springer, Berlin, 2018), arXiv:1805.06422

Some key papers:
M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, Phys. Rev. Lett. 100, 030602 (2008).
A. J. Short and T. C. Farrelly, New J. Phys. 14, 013063 (2012)
Sheldon Goldstein, Takashi Hara, and Hal Tasaki Phys. Rev. Lett. 111, 140401 (2013)
L. P. García-Pintos, N. Linden, A. S. L. Malabarba, A. J. Short, and A. Winter, Phys. Rev. X 7, 031027 (2017).
P. Reimann, Nature Comm. 7, 10821 (2016)

Equilibration times: A very brief overview

- **Numerically and empricially**, one usually finds that equilibration happens **relatively quickly**. (In times at most polynomial in the system size. This is required, due to **finite-group velocity,** i.e., Lieb-Robinson bounds.)



- In **non-interacting fermionic** and **bosonic** systems, equilibration times can be proven. Initially **homogeneous states** typically equilibrate locally according to **power-law.**

- **Interacting systems:** Extremely difficult problem to get rigorous and meaningful results.

Equilibration times: A general result

A. J. Short and T. C. Farrelly, New J. Phys. 14, 013063 (2012), arXiv:1110.5759 .

General bound:

$$\overline{(\mathrm{Tr}[A\rho(t)] - \mathrm{Tr}[A\overline{\rho}])^2}^T \leq \|A\|^2 D_G \mathrm{e}^{-S_2(\vec{p}_E)} \frac{5\pi}{2} \left(\frac{3}{2} + \frac{1}{\epsilon T} \right)$$

$\epsilon :=$ Minimal difference between energy gaps.

Problem: In many-body systems, $\epsilon \sim \exp(-N)$. Thus, T ~ exp(N): much smaller than recurrence time, but still unrealistically big for physical observables.

A more promising bound is shown in L. P. García-Pintos, N. Linden, A. S. L. Malabarba, A. J. Short, and A. Winter, Phys. Rev. X 7, 031027 (2017) under additional assumptions. These assumptions seem to be natural, but are difficult to explicitly show to hold.

→ **Showing reasonable equilibration times from physically clear and sensible assumptions for interacting systems is still a major open problem.**

Summary

Equilibration means that

Infinite time-average

$$\langle\Psi(t)|A|\Psi(t)\rangle \approx \mathrm{Tr}[A\overline{\rho}], \quad \overline{\rho} = \overline{|\Psi(t)\rangle\langle\Psi(t)|}^{\infty}$$

for “physically relevant” observables and most times.

Thermalization means that additionally $\mathrm{Tr}[A\overline{\rho}] \approx \mathrm{Tr}[A\mathrm{e}^{-\beta H}/Z_\beta]$

- **Equilibration** for local observables **generic in infinite time** if few degenerate energy gaps present in system.
  Large amounts of **entanglement** in energy eigenstates beneficial.

- **Thermalization** if entropy density large enough, or ETH holds.
  (Both difficult to prove: Big open problem!)

- **Equilibration times** usually expected to be short, but notoriously difficult to prove in general. Big open problem!

Some references (incomplete and biased)

Reviews:

- Nature Physics 11, 124 (2015) Eisert, Friesdorf, Gogolin
- Advances in Physics 65, 3 (2016) D'Alessio, Kafri, Polkovnikov, Rigol
- Rep. Prog. Phys. 79, 056001 (2016) Gogolin, Eisert

For recent pedagogical explanation, see for example:

- H.W., T. R. de Oliveira, A. J. Short, J. Eisert, in "Thermodynamics in the quantum regime", F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso (eds.) (Springer, Berlin, 2018)

Equilibration and equilibration times:

- P. Reimann, Phys. Rev. Lett. 101, 190403 (2008), arXiv:0810.3092 .
- A. J. Short and T. C. Farrelly, New J. Phys. 14, 013063 (2012), arXiv:1110.5759
- H.W., M. Goihl, I. Roth, J. Eisert, Phys. Rev. Lett. 123, 200604 (2019)
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Thermalization:

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- M. P. Mueller, E. Adlam, L. Masanes, N. Wiebe, Communications in Mathematical Physics 340 2 (2015)
- T. Farrelly, F. G. S. L. Brandao, M. Cramer, Phys. Rev. Lett. 118, 140601 (2017)