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Asymptotic reversibility of thermal operations for interacting quantum spin systems

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Bad Honnef, Feb. 2020

When does thermodynamic behavior emerge in many-body systems?

- Simplified description of the physical behavior of a system, no need to know microscopic details

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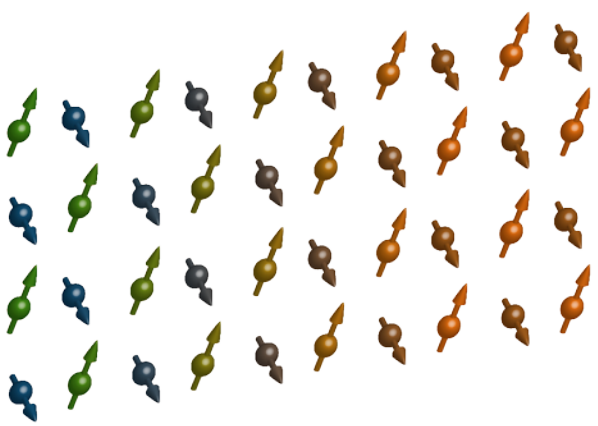

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steady state, no memory of initial conditions

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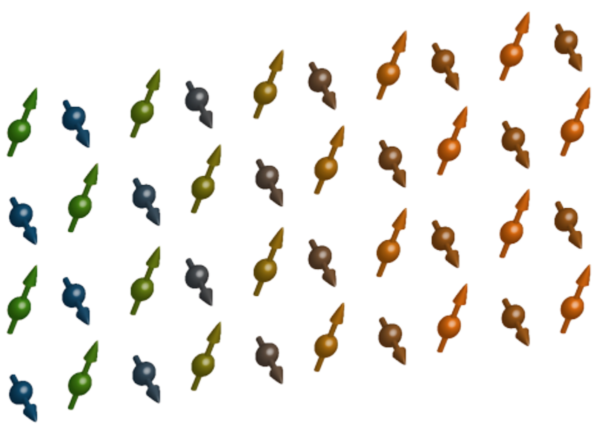
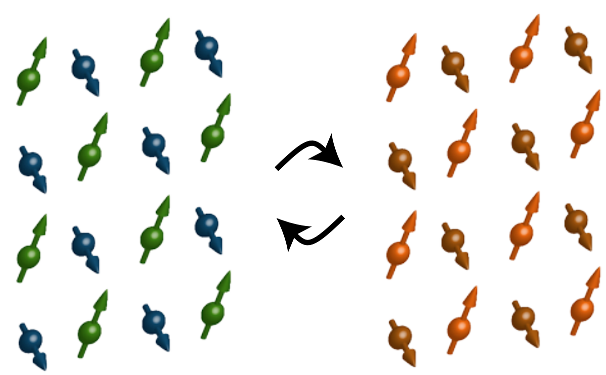
reversible processes





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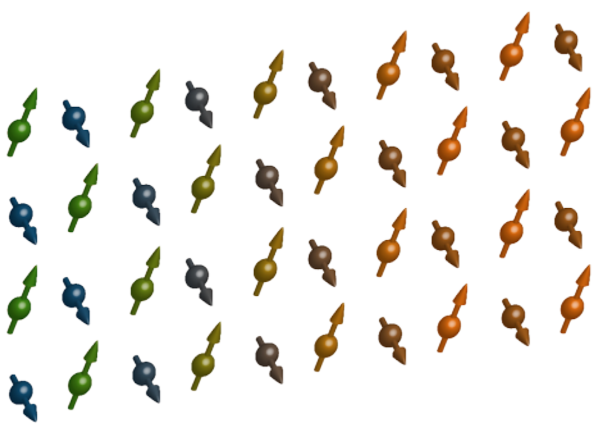

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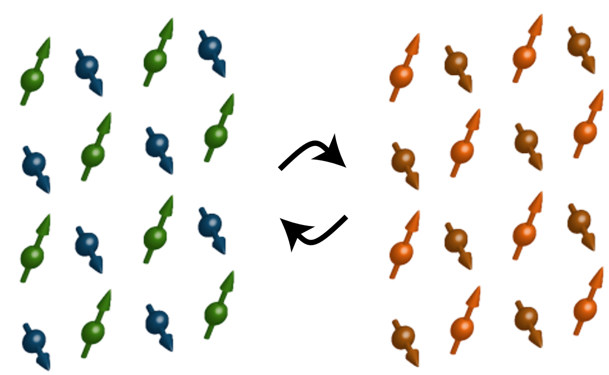

→ thermodynamic
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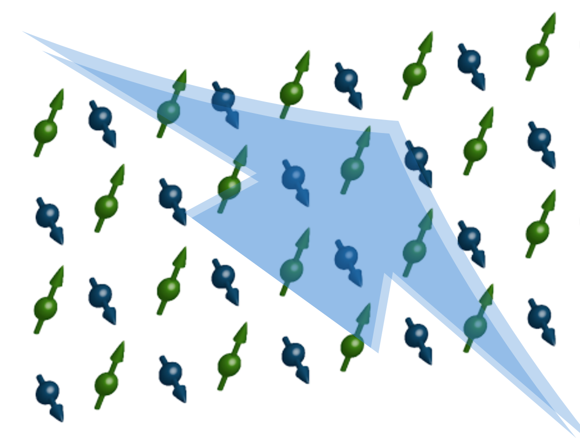

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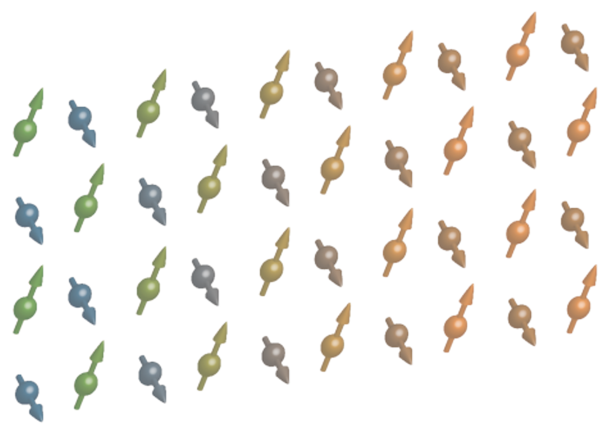

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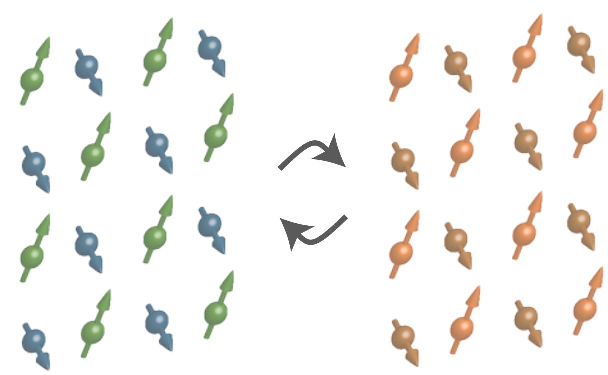

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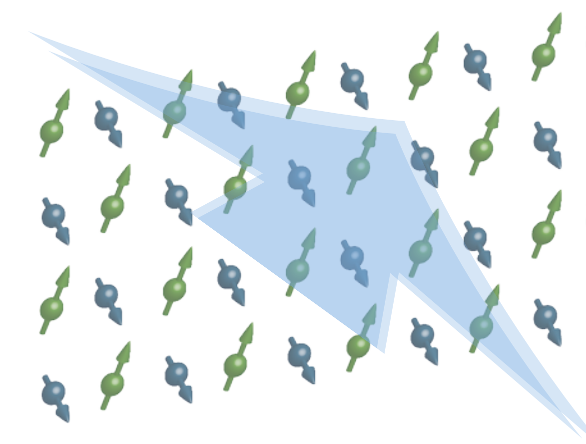

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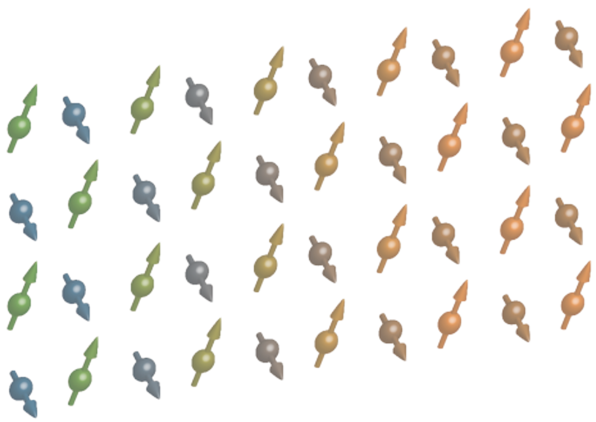

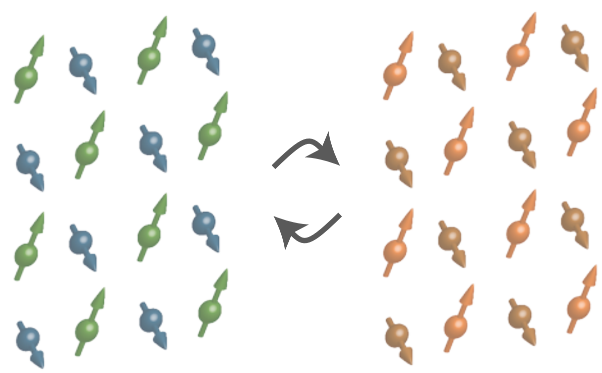

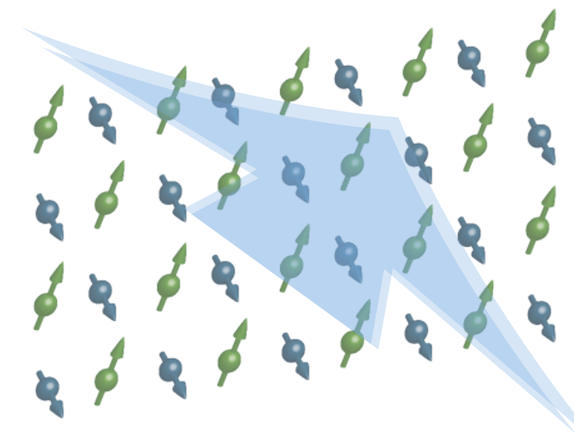

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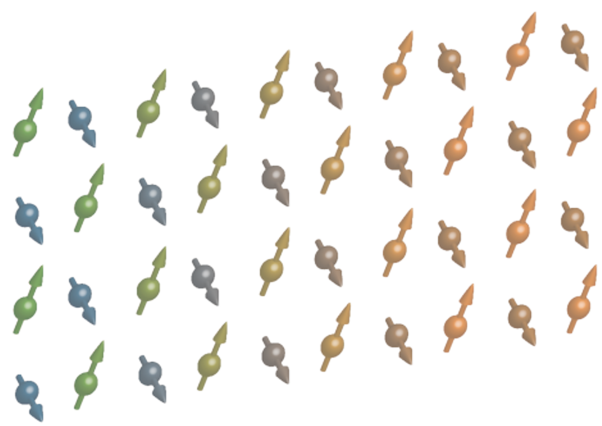

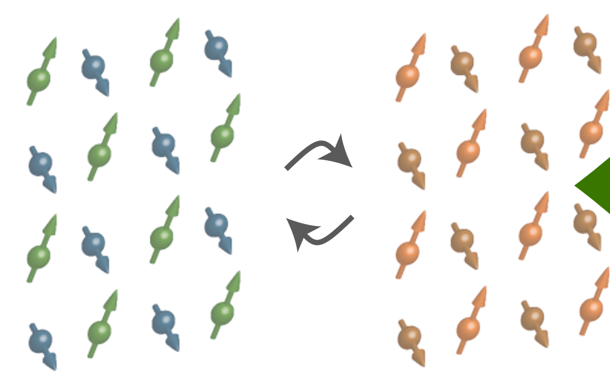

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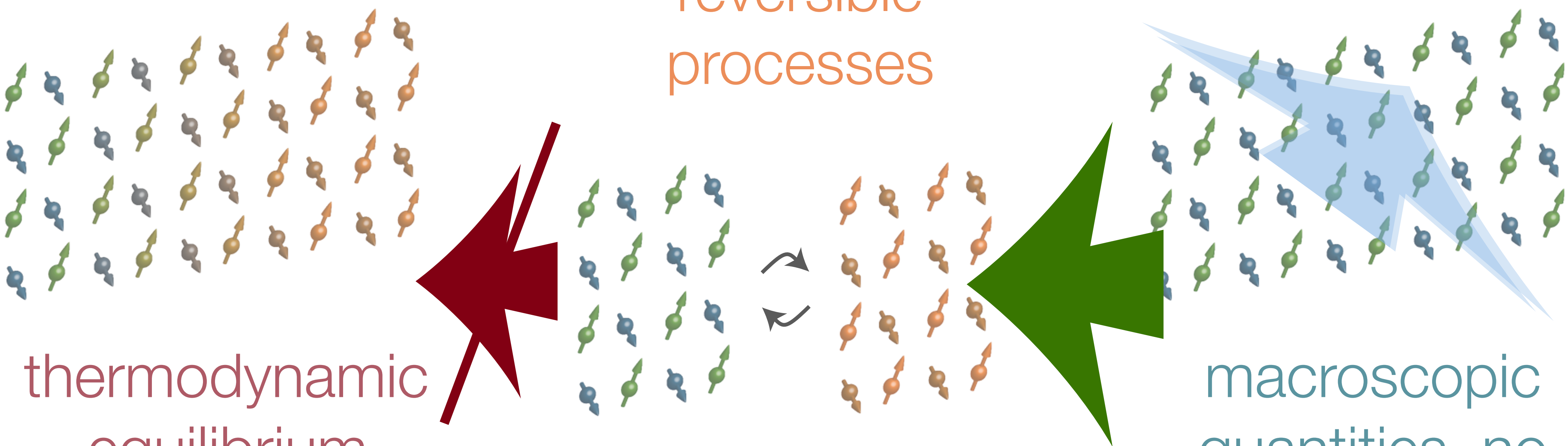

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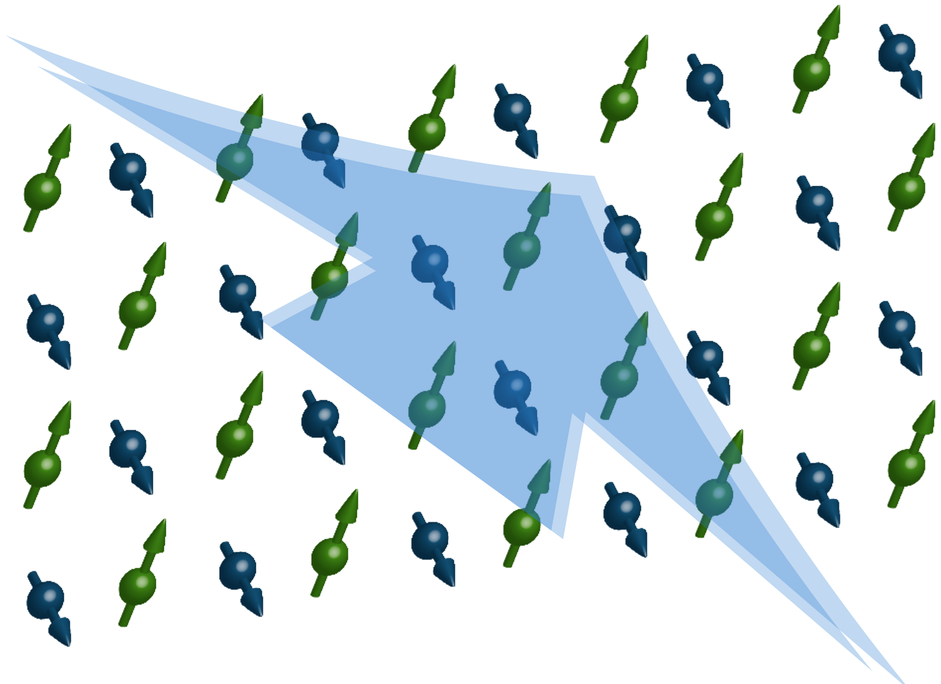
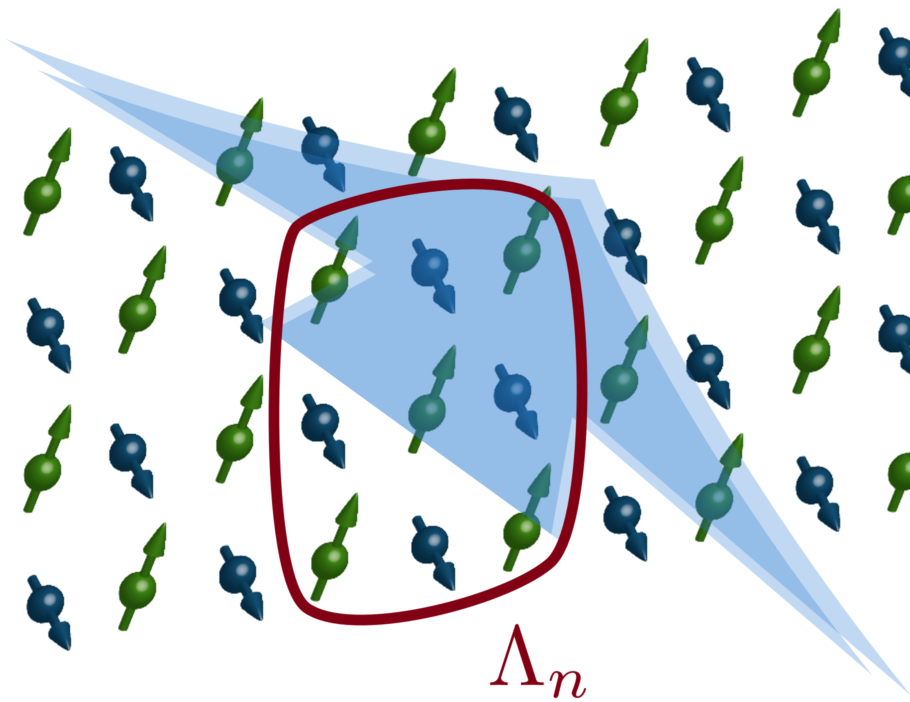
Quantum ergodicity



- All translation-invariant observables must have vanishing fluctuations

Ruelle 1999; Bjelaković *et al.* CMP 2004

Quantum ergodicity



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ρ is ergodic if for all a_0,

$$\mathrm{Var}_\rho\left(\frac{1}{n}\sum_{i\in\Lambda_n} a_i\right) \to 0$$

a_0 shifted by i

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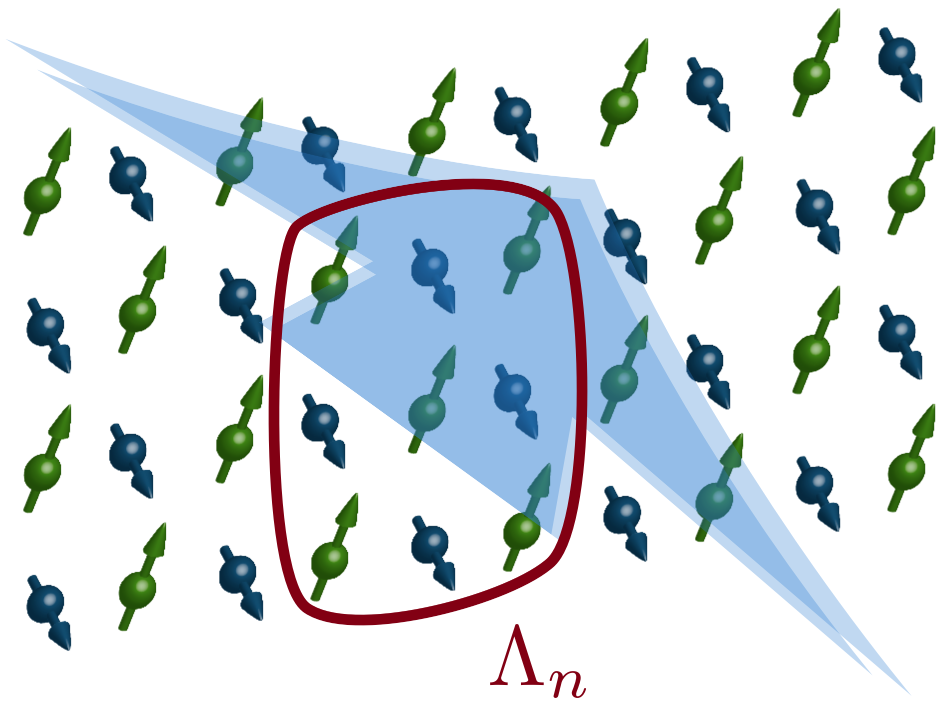

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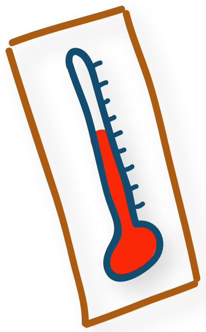
- equivalently: ρ is ergodic if it is extremal in the set of translation-invariant states

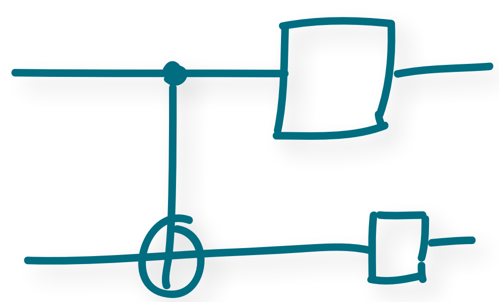
Ruelle 1999; Bjelaković *et al.* CMP 2004

Resource theory of thermal operations



Allowed ancillas in their thermal state

Allowed energy-conserving unitaries



Allowed to discard any system

Brandão+ PRL 2013; Horodecki & Oppenheim Nat Com 2013; Ng+ PNAS 2015; Chitambar & Gour RMP 2019; …

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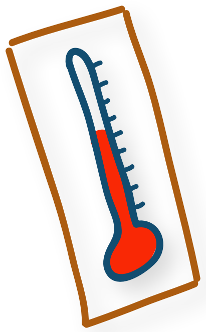

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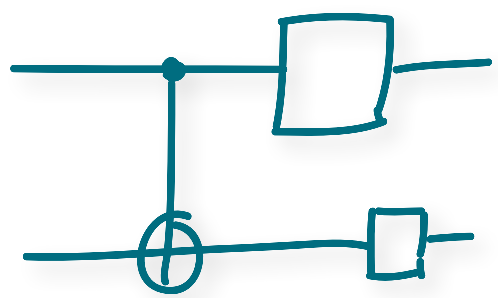

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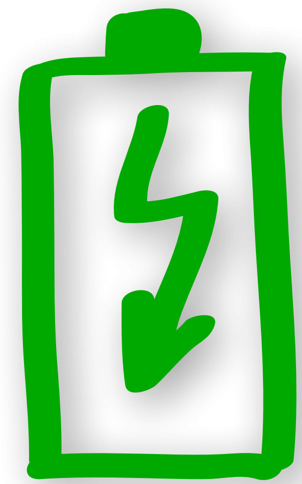


Account for work using a battery

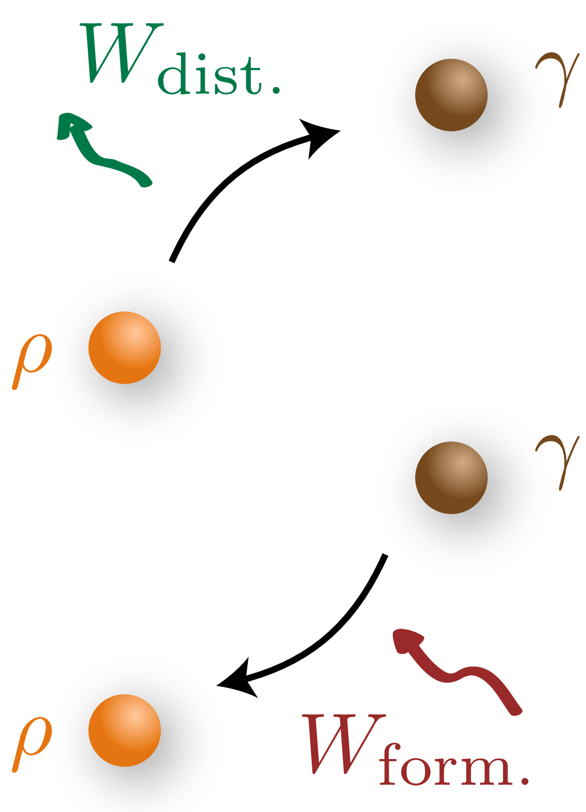
$$\rho \otimes |E\rangle\langle E| \to \rho' \otimes |E'\rangle\langle E'|$$

"consumes $E - E'$ work"

Brandão+ PRL 2013; Horodecki & Oppenheim Nat Com 2013; Ng+ PNAS 2015; Chitambar & Gour RMP 2019; …

Work distillation & state formation

for semiclassical states (block-diagonal in energy)



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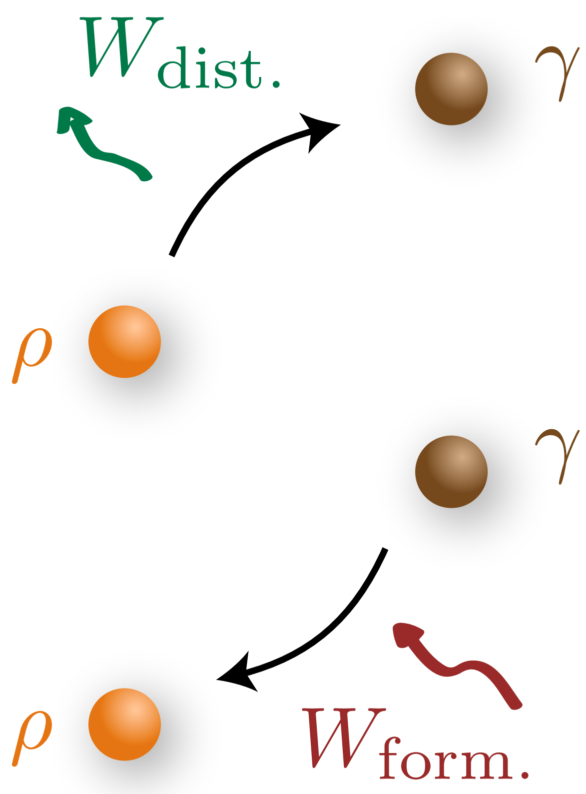

$$\beta W_{\text{dist.}} \boxed{= S^{\epsilon}_{\min}(\rho \,\|\, \gamma) = \max_{\tilde\rho \approx \rho}\left[-\log \operatorname{tr}\left(\Pi^{\tilde\rho}\gamma\right)\right]}$$

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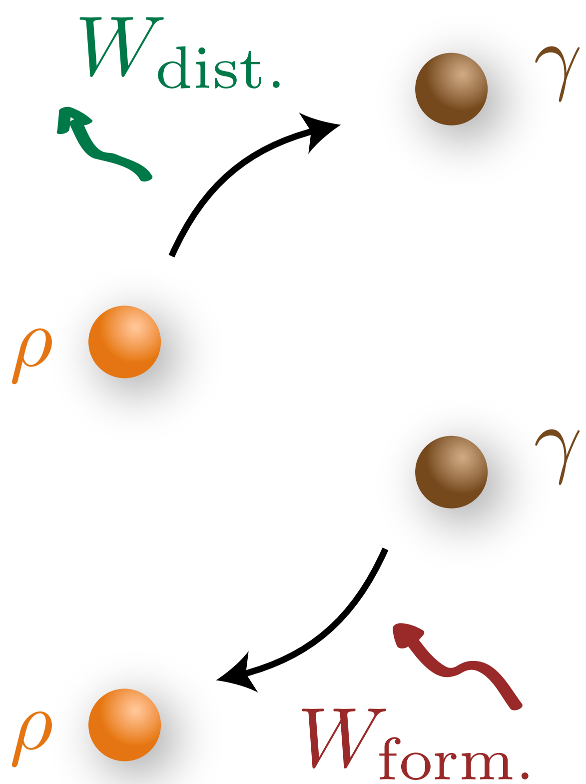

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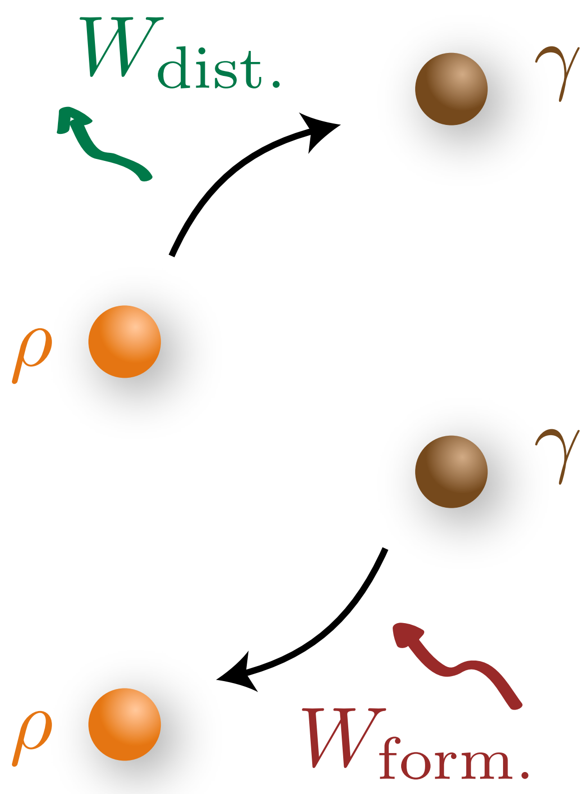

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- reversible if $S_{\min} = S_{\max}$

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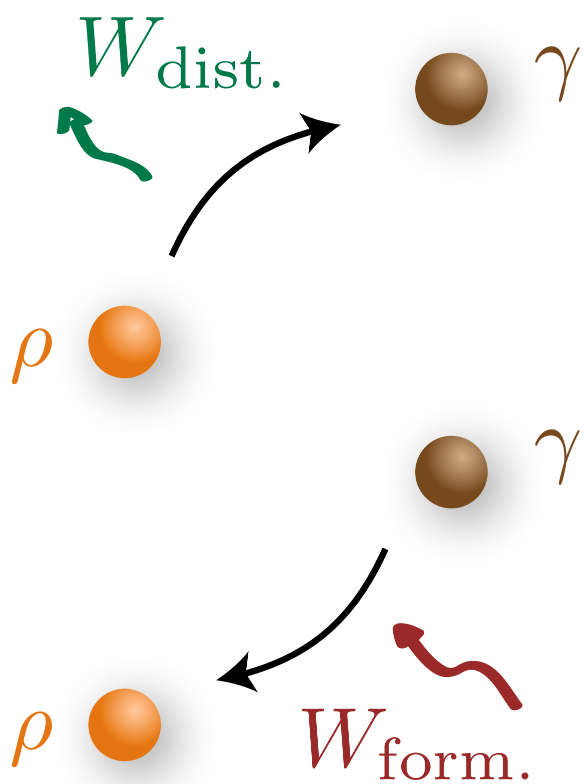

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- reversible if $S_{\min} = S_{\max}$
- does not apply to fully quantum states

Åberg, N Comm 2013;
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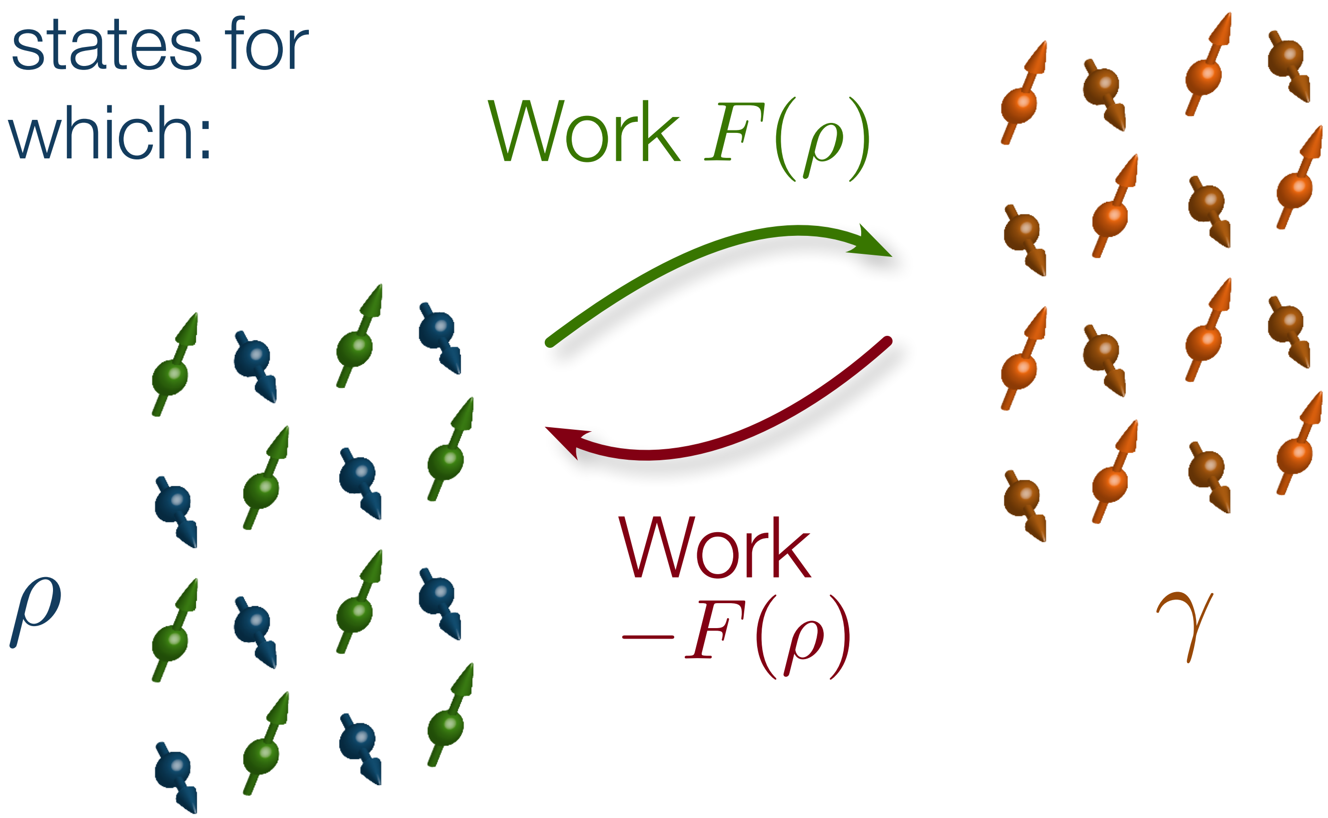
The thermodynamic potential

class of
states for
which:



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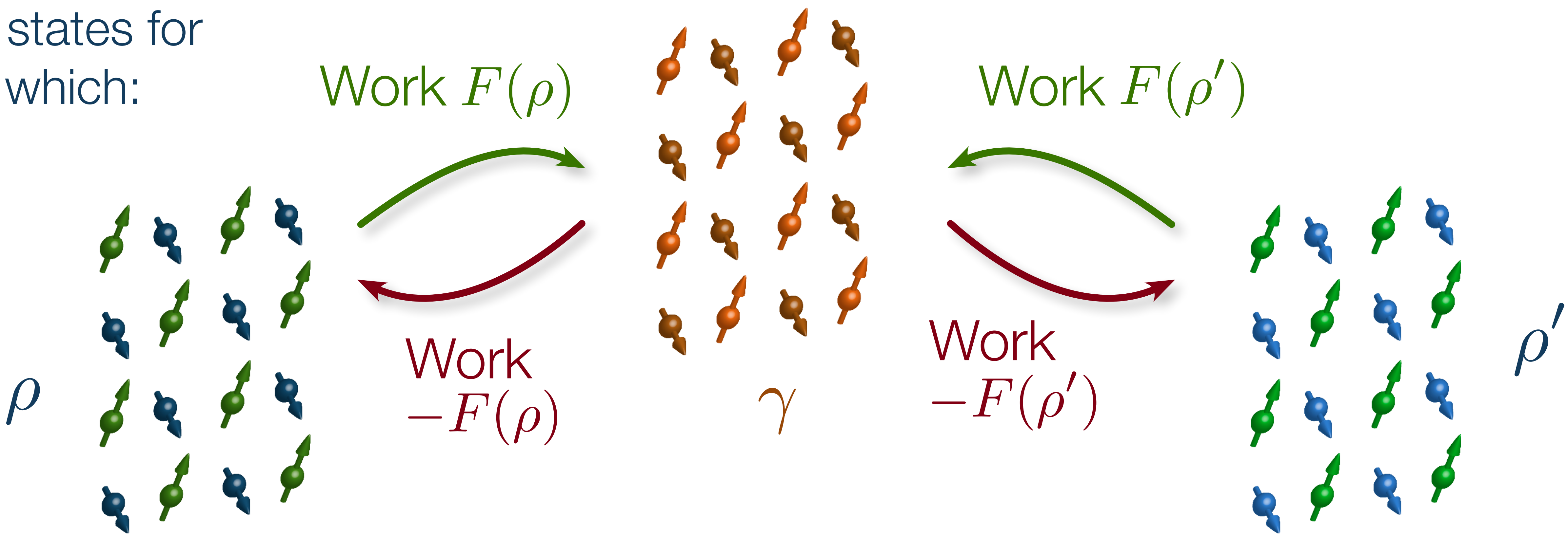

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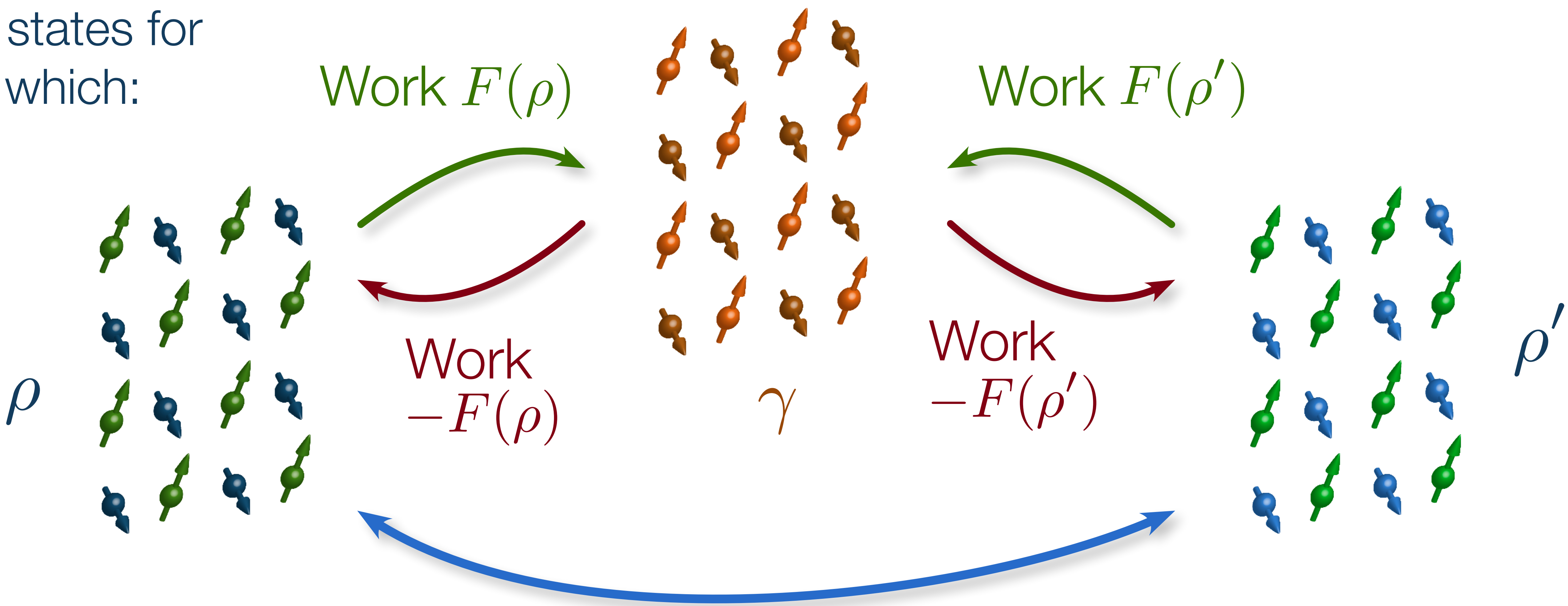

Reversible cost $F(\rho') - F(\rho)$
derives from thermodynamic potential

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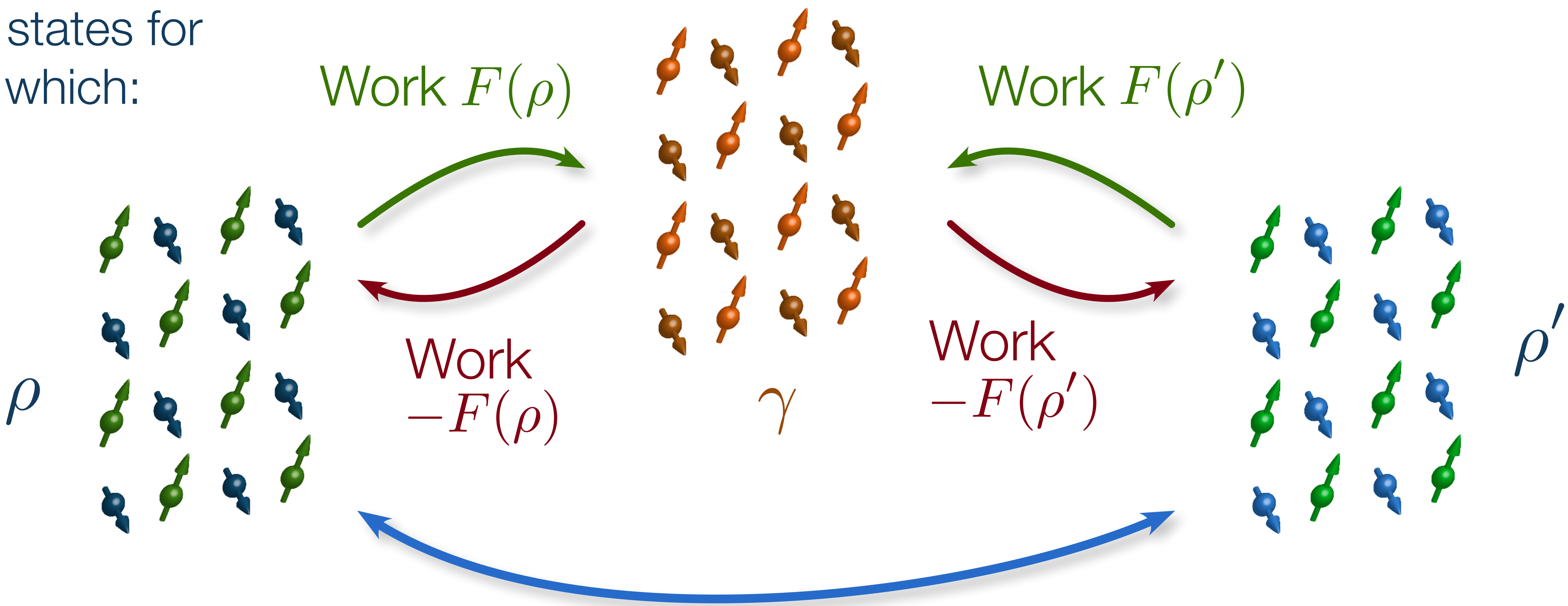

Reversible cost $F(\rho') - F(\rho)$
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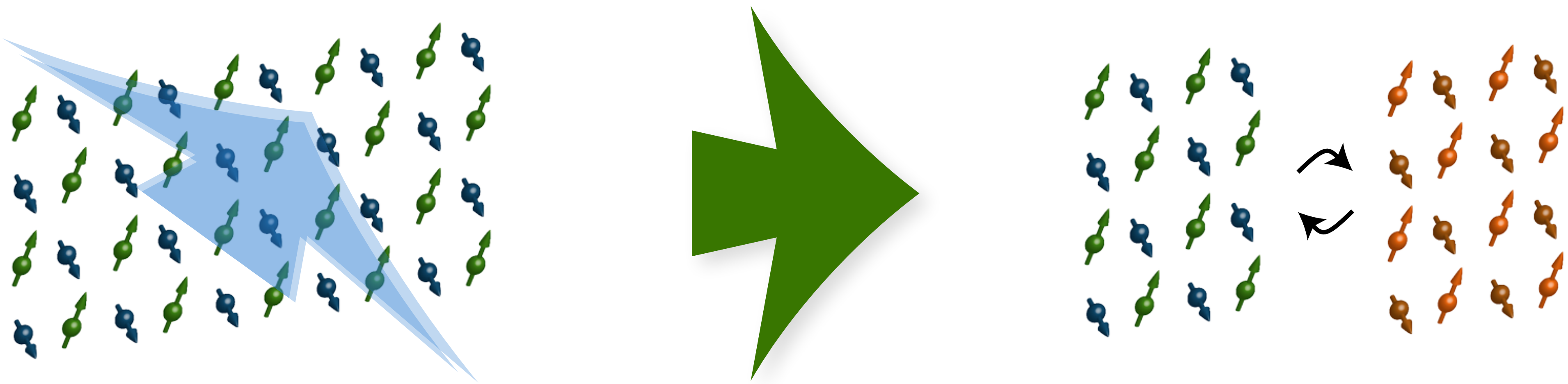
- **i.i.d. states**
- **statistical ensembles**
- **other, perhaps more realistic settings?**

Main Result: ergodicity implies reversibility



*

ρ, ρ' ergodic
local Hamiltonian

$$\frac{1}{n}\,\mathrm{Work}(\rho \to \rho') \xrightarrow{n\to\infty} \beta^{-1}[s(\rho') - s(\rho)]$$

emergent
thermodynamic
potential

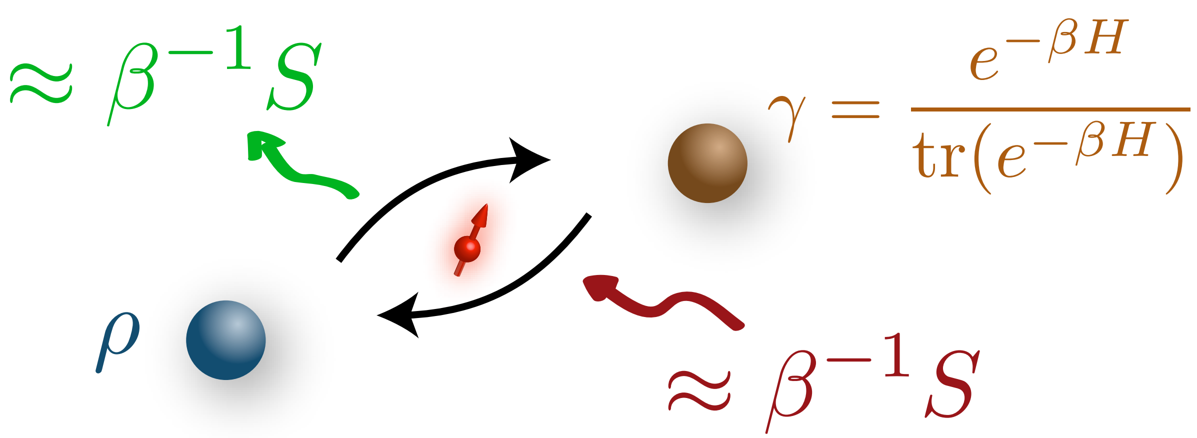
$$s(\rho) = \lim_{n\to\infty} \tfrac{1}{n} S(\rho_n \,\|\, e^{-\beta H_n})$$

**Terms and conditions apply.*

Proof in 2 main steps

- #1: Criterion for reversibility in terms of min- and max-relative entropies

$$\left.\begin{array}{r} S^{\epsilon}_{\min}(\rho\,\|\,\gamma) \\ S^{\epsilon}_{\max}(\rho\,\|\,\gamma) \end{array}\right\} \approx S \quad \Rightarrow$$



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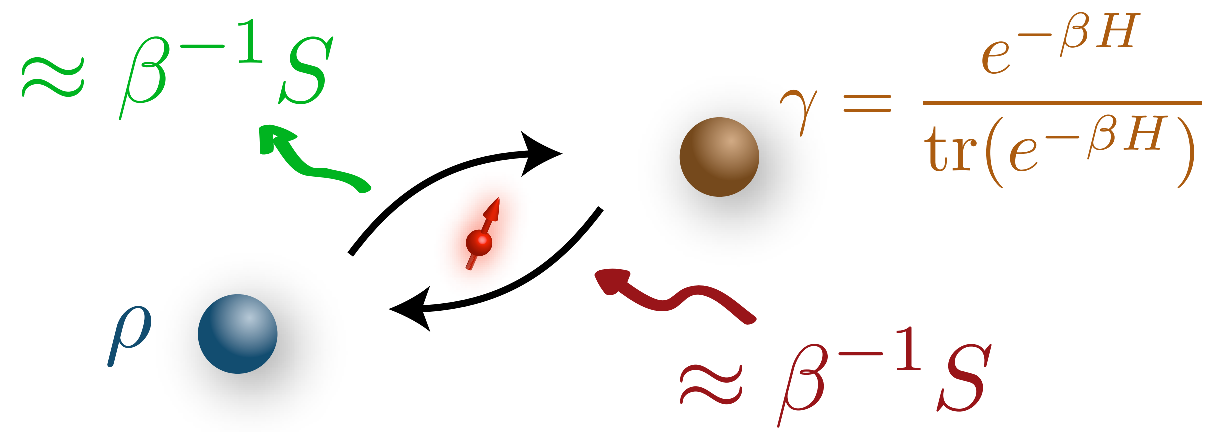

- #2: Criterion satisfied for ergodic states & local Hamiltonians

new Stein's lemma for ergodic states & local Gibbs states

$$\left.\begin{matrix} \frac{1}{n} S^{\epsilon}_{\min}(\rho_n \,\|\, e^{-\beta H_n}) \\ \frac{1}{n} S^{\epsilon}_{\max}(\rho_n \,\|\, e^{-\beta H_n}) \end{matrix}\right\} \xrightarrow{n\to\infty} s(\rho) = \lim \frac{1}{n} S(\rho_n \,\|\, e^{-\beta H_n})$$

$$[\, S_{\ldots}(\rho \,\|\, \gamma) = S_{\ldots}(\rho \,\|\, e^{-\beta H}) + \log \mathrm{tr}(e^{-\beta H}) \,]$$

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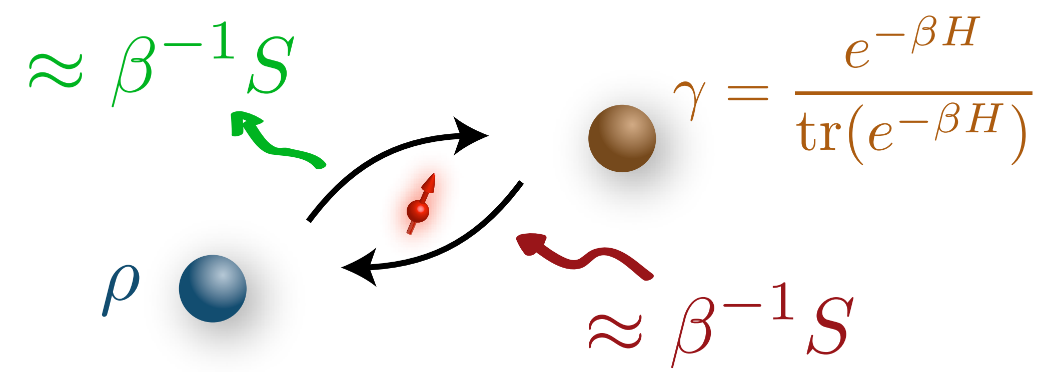

- #2 $\left.\begin{matrix} \frac{1}{n} S^{\epsilon}_{\min}(\rho_n \,\|\, e^{-\beta H_n}) \\ \frac{1}{n} S^{\epsilon}_{\max}(\rho_n \,\|\, e^{-\beta H_n}) \end{matrix}\right\} \xrightarrow{n\to\infty} s(\rho) = \lim \frac{1}{n} S(\rho_n \,\|\, e^{-\beta H_n})$

Given any two ergodic states ρ, ρ'

$$[\, S_{\dots}(\rho \,\|\, \gamma) = S_{\dots}(\rho \,\|\, e^{-\beta H}) + \log \mathrm{tr}(e^{-\beta H}) \,]$$

Proof in 2 main steps

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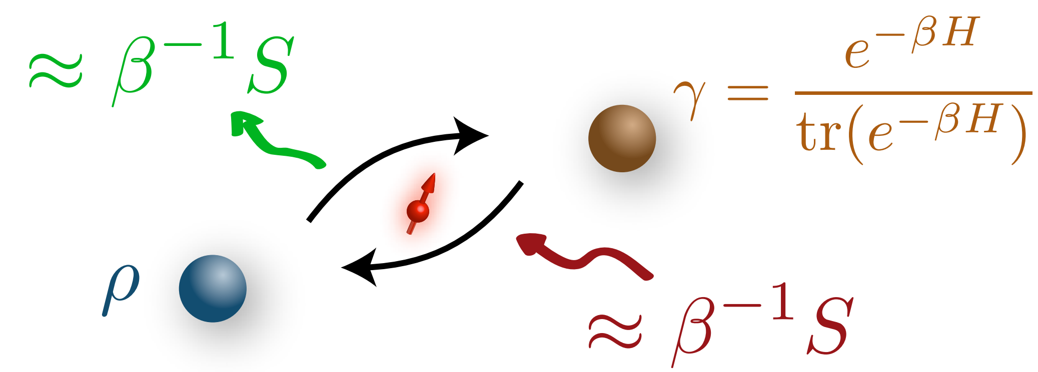

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Given any two ergodic states ρ, ρ' $\xrightarrow{\#2}$ min and max relative entropy collapse to a single value s(ρ), s(ρ')

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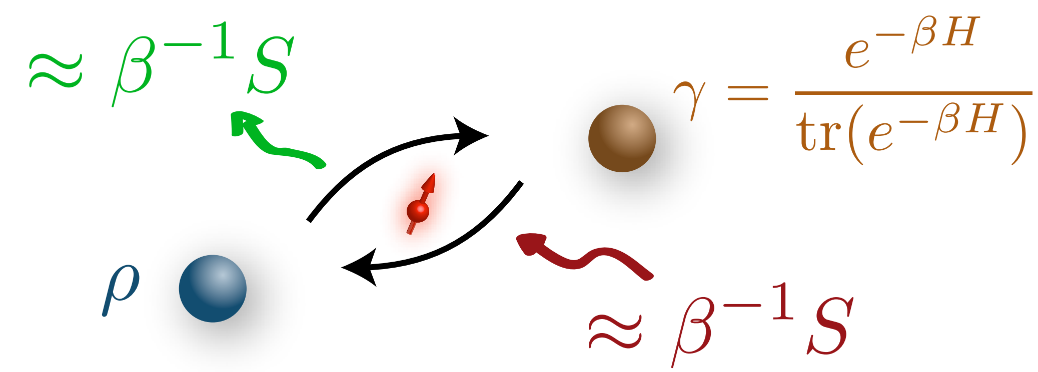

ρ and ρ' can each be approx. converted reversibly to/from γ

#1

- #2 $\left.\begin{array}{l} \frac{1}{n} S^{\epsilon}_{\min}(\rho_n \,\|\, e^{-\beta H_n}) \\ \frac{1}{n} S^{\epsilon}_{\max}(\rho_n \,\|\, e^{-\beta H_n}) \end{array}\right\} \xrightarrow{n\to\infty} s(\rho) = \lim \frac{1}{n} S(\rho_n \,\|\, e^{-\beta H_n})$

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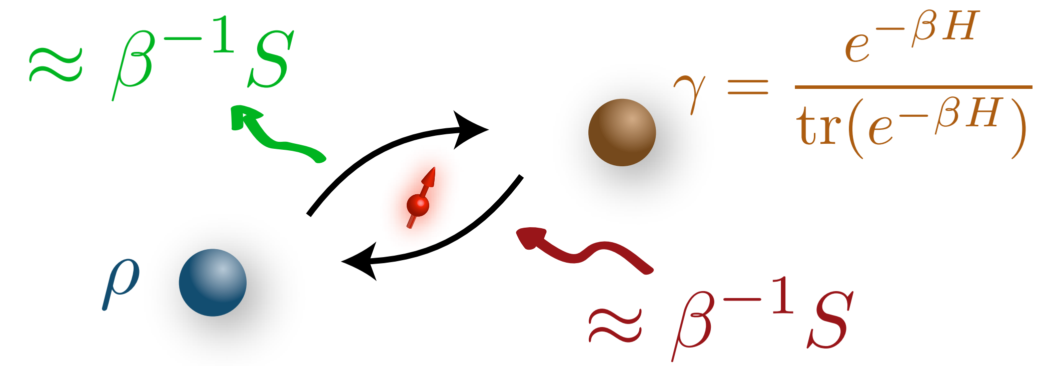

ρ can be reversibly converted to/from ρ' in the thermodynamic limit ← ρ and ρ' can each be approx. converted reversibly to/from γ ← #1

- #2 $\left.\begin{array}{l} \frac{1}{n} S^{\epsilon}_{\min}(\rho_n \,\|\, e^{-\beta H_n}) \\ \frac{1}{n} S^{\epsilon}_{\max}(\rho_n \,\|\, e^{-\beta H_n}) \end{array}\right\} \xrightarrow{n\to\infty} s(\rho) = \lim \frac{1}{n} S(\rho_n \,\|\, e^{-\beta H_n})$

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$[\, S_{\dots}(\rho \,\|\, \gamma) = S_{\dots}(\rho \,\|\, e^{-\beta H}) + \log \mathrm{tr}(e^{-\beta H}) \,]$

Some technical proof ingredients

- Coherence modes of quantum states

  Korzekwa *et al.*, NJP (2016); Marvian & Spekkens, PRA (2014)

- Small reference frame for creating coherent superpositions of energy levels

  Bartlett *et al.*, RMP (2007); Brandão *et al.*, PRL (2013); …

- Typical subspaces for ergodic states & local Gibbs states

  Bjelaković & Siegmund-Schultze, CMP (2004)

Thermodynamic behavior w/o equilibrium

- Ergodic states are spatially ergodic (w.r.t. translation-invariant observables) and can evolve nontrivially in time

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e.g. $|\psi\rangle = |+\rangle^{\otimes n}$

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$|+\rangle = [|\uparrow\rangle + |\downarrow\rangle]/\sqrt{2}$

$H = \sum \sigma_z$

Thermodynamic behavior w/o equilibrium

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Thermodynamic behavior w/o equilibrium

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$H = \sum \sigma_z$

i.i.d. $\Rightarrow$ ergodic

  - $|\psi\rangle$ has macroscopic changes in $\langle \sum \sigma_x \rangle$

- and yet such states are asymptotically reversibly convertible with thermal operations

Discussion

- New Shannon-McMillan-Breiman theorem for the relative entropy
- New result for thermal operations that applies to states with coherences
- Small reference frame can be a classical field / laser light
- local reduced state of Gibbs instead of truncated Hamiltonian, for high enough T
- The KL divergence is not always the relevant potential (but it is for ergodic states)

Outlook

- What is the largest class of states that obey the reversibility property?
- Relation to entropy accumulation?

Dupuis *et al.*, 1607.01796

- Can we incorporate small violations of translation-invariance? or some disorder?
- Any connections to the eigenstate thermalization hypothesis? (Hint: exponential decay of off-diagonal matrix elements)

Thank you for
your attention!

#1 Criterion for approximate reversibility

$$\begin{cases} S^{\epsilon}_{\min}(\rho \,\|\, e^{-\beta H}) \geqslant S - \Delta \\ S^{\epsilon}_{\max}(\rho \,\|\, e^{-\beta H}) \leqslant S + \Delta \end{cases}$$

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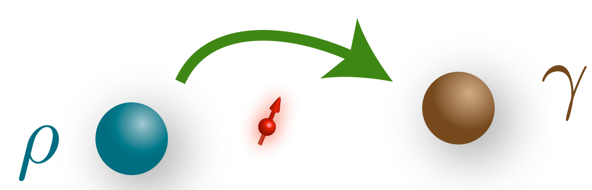
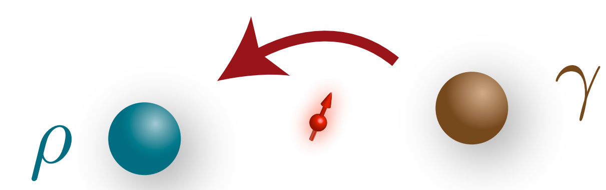
Protocols for 

& 

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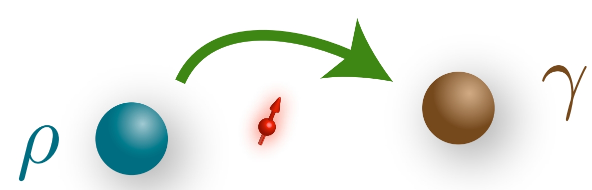
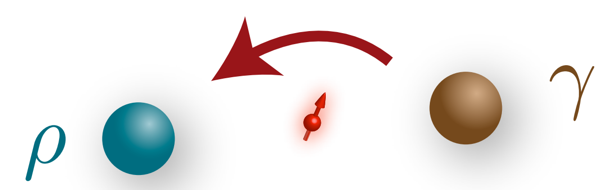
Protocols for 

& 

:

- energies multiple of $O(\Delta)$
- dephase in energy spaces
- semiclassical work extraction

Work $\geqslant \beta^{-1} S - O(\Delta)$

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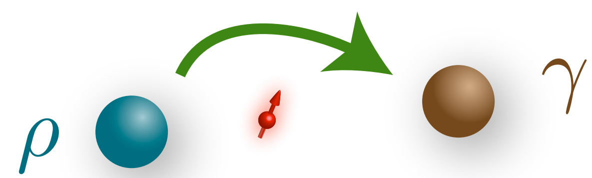
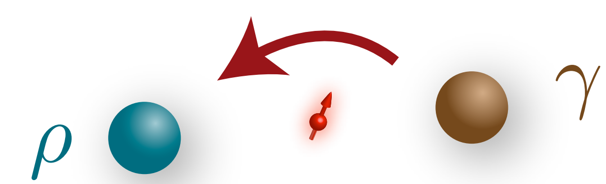
Protocols for 

& 

:

- energies multiple of $O(\Delta)$
- dephase in energy spaces
- semiclassical work extraction

Work $\geqslant \beta^{-1} S - O(\Delta)$

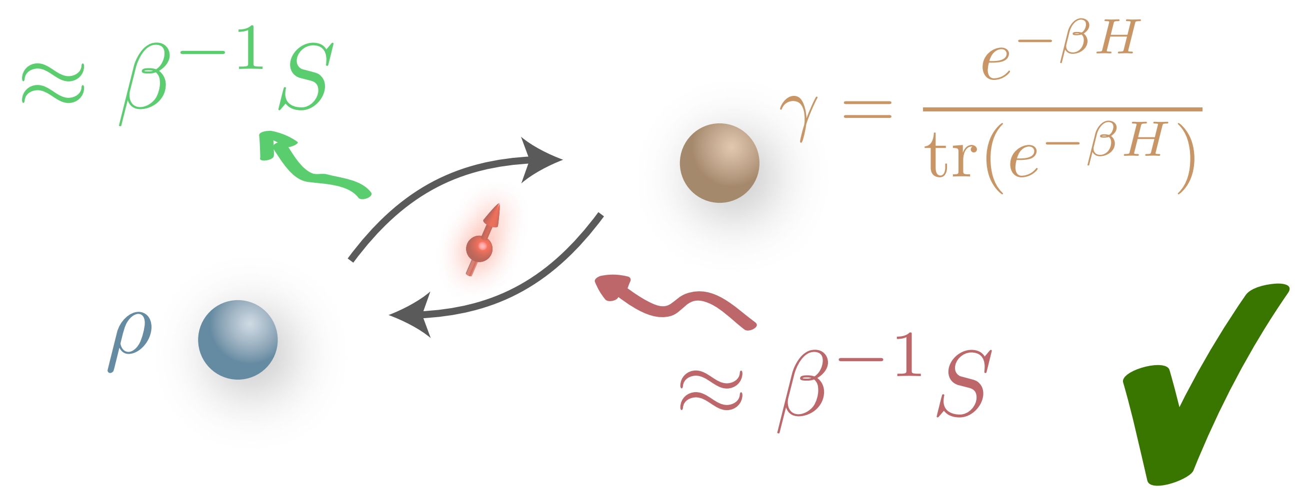
- energies multiple of $O(\Delta)$
- create ρ & reference frame (semiclassical formation)
- shift coherence to ρ

Work $\leqslant \beta^{-1} S + O(\Delta)$

Proof in 2 main steps

- #1: Criterion for reversibility in terms of min- and max-relative entropies

$$\left.\begin{array}{l} S^{\epsilon}_{\min}(\rho \,\|\, \gamma) \\ S^{\epsilon}_{\max}(\rho \,\|\, \gamma) \end{array}\right\} \approx S \quad \Rightarrow$$



- #2: Criterion satisfied for ergodic states & local Hamiltonians

new Stein's lemma for ergodic states & local Gibbs states

$$\left.\begin{array}{l} \frac{1}{n} S^{\epsilon}_{\min}(\rho_n \,\|\, e^{-\beta H_n}) \\ \frac{1}{n} S^{\epsilon}_{\max}(\rho_n \,\|\, e^{-\beta H_n}) \end{array}\right\} \xrightarrow{n\to\infty} s(\rho) = \lim \frac{1}{n} S(\rho_n \,\|\, e^{-\beta H_n})$$

Stein's lemma

The hypothesis testing relative entropy

$$S_{\mathrm{h}}^{\eta}(\rho \,\|\, \sigma) = -\log \min_{\substack{0 \leqslant Q \leqslant \mathbb{1} \\ \mathrm{tr}(Q\rho) \geqslant \eta}} \mathrm{tr}(Q\sigma)$$

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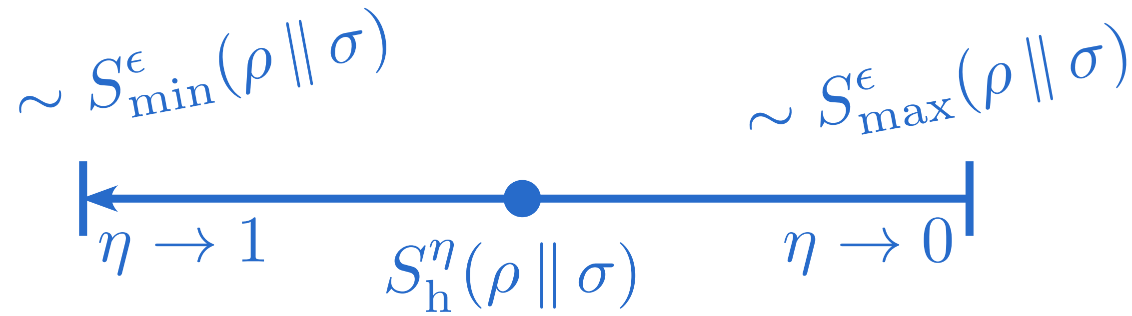
interpolates between $S^{\epsilon}_{\mathrm{min}}$ and $S^{\epsilon}_{\mathrm{max}}$



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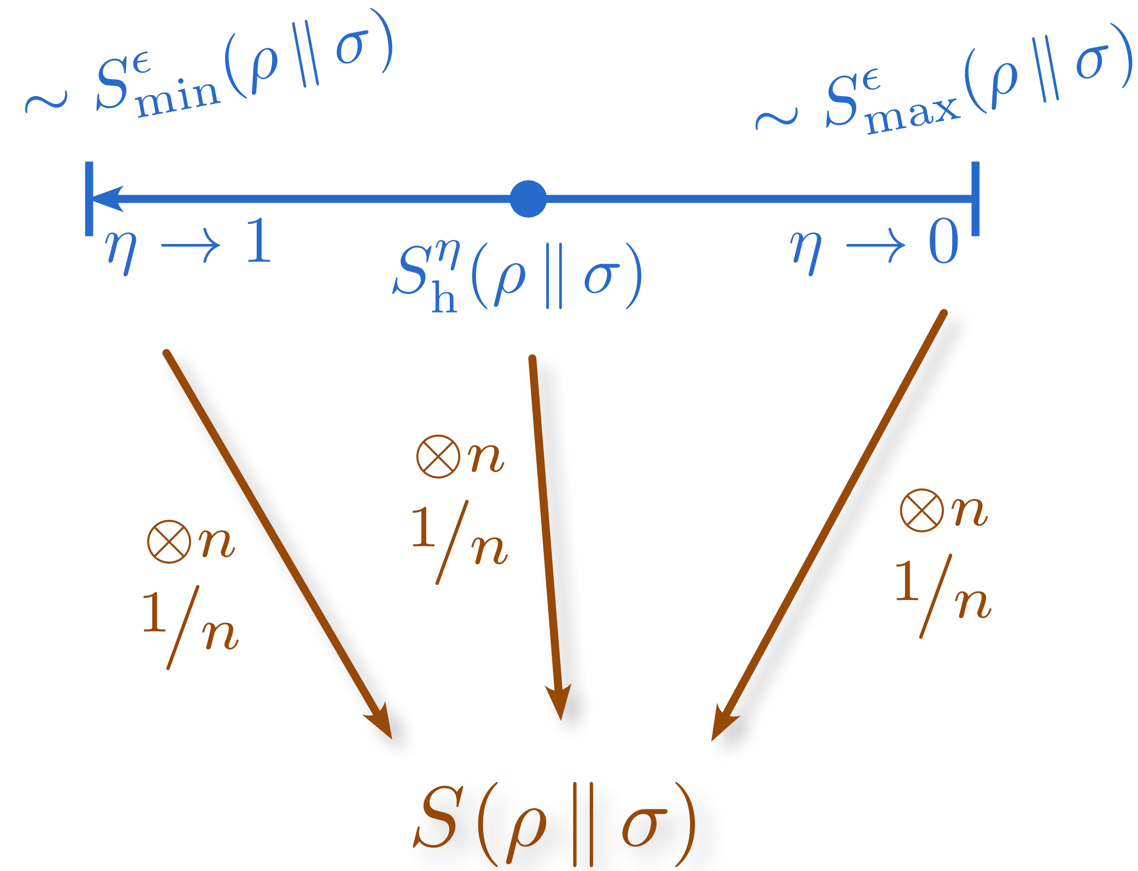
Stein's lemma for i.i.d. states



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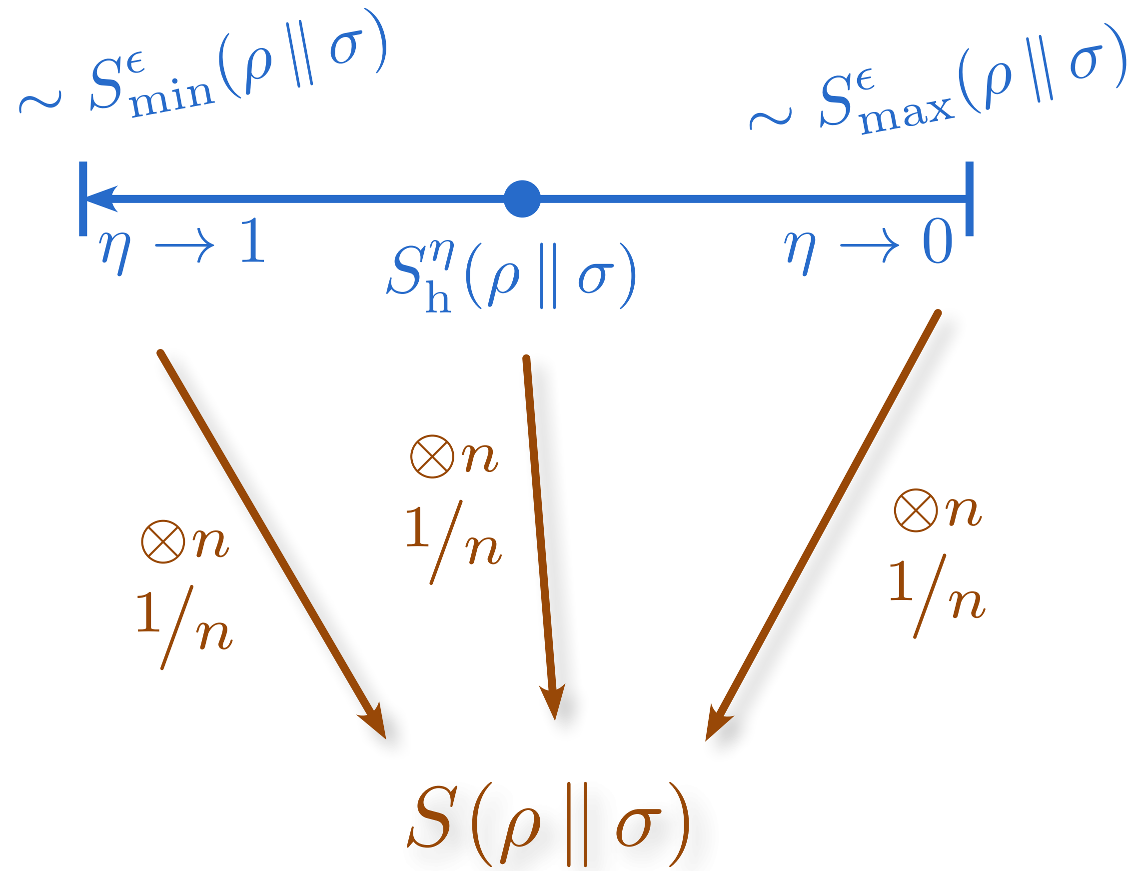
Stein's lemma for i.i.d. states



also:

- ergodic states & product states
- classically: Shannon-McMillan-Breiman

Bjelaković *et al.* CMP 2004, Cover & Thomas

Stein's lemma (i.i.d.)

on site i

$$S_{\mathrm{h}}^{\eta}(\rho^{\otimes n} \,\|\, e^{-H_n}) = \,?$$

$$e^{-H_n} = \sigma^{\otimes n}$$

$$H_n = \sum h_i$$

$$h_i = h = -\ln(\sigma)$$

eigenstates $|x\rangle$

Bjelaković+ quant-ph/0307170

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$$\mathrm{tr}\big(\Pi_{\rho|\sigma}^{n,\delta}\rho^{\otimes n}\big) \to 1$$ (large deviations)

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$$\leqslant e^{-n(\langle h\rangle_\rho - \delta)}\,\mathrm{tr}\big(\Pi_{\rho|\rho}^{n,\delta}\big)$$

$$\leqslant e^{-n(\underbrace{\langle h\rangle_\rho - S(\rho)}_{S(\rho\,\|\,\sigma)} - 2\delta)}$$

use

$$Q = \Pi_{\rho|\sigma}^{n,\delta}\Pi_{\rho|\rho}^{n,\delta}\Pi_{\rho|\sigma}^{n,\delta}$$

$\to$ 1

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(converse bound via SDP dual)

Stein's lemma (ergodic & local Gibbs)

$S_{\mathrm{h}}^{\eta}(\ \rho_n \ \| \, e^{-H_n}) = \, ?$ $\sigma_n = e^{-H_n}$ ρ_n ergodic; H_n local with eigenstates $|x^n\rangle$

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$$\mathrm{tr}\big(\Pi^{n,\delta}_{\rho|\sigma}\,\rho_n\big) \to 1 \quad \text{(ergodicity)}$$

candidate in

$$e^{-S_{\mathrm{h}}^{\eta}(\,\rho_n\,\|\,\sigma_n\,)} = \min_{\substack{0\leqslant Q\leqslant \mathbb{1}\\ \mathrm{tr}(Q\,\rho_n\,)\geqslant\eta}} \mathrm{tr}(Q\,\sigma_n\,)\quad ?$$

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$$\leqslant e^{-n(\langle \frac{1}{n}H_n\rangle_\rho-\delta)}\,\mathrm{tr}(\Pi^{n,\delta}_{\rho|\rho})$$

$$\leqslant e^{-n(s_n-2\delta\,)}$$

use

$$Q = \Pi^{n,\delta}_{\rho|\sigma}\Pi^{n,\delta}_{\rho|\rho}\Pi^{n,\delta}_{\rho|\sigma} \longrightarrow 1$$

where $s_n \to \lim \frac{1}{n} S(\rho_n\,\|\,\sigma_n)$

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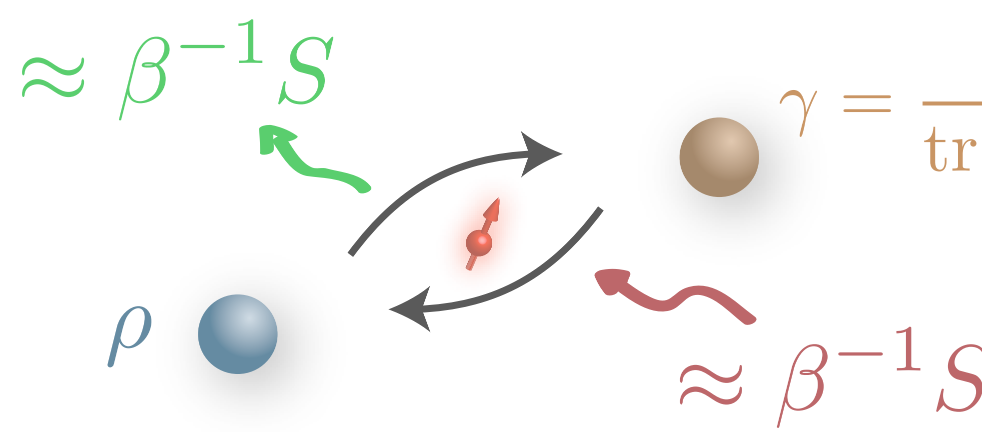
(converse bound via SDP dual)

Proof in 2 main steps

- #1: Criterion for reversibility in terms of min- and max-relative entropies

$$\left.\begin{array}{l} S^{\epsilon}_{\min}(\rho\,\|\,\gamma) \\ S^{\epsilon}_{\max}(\rho\,\|\,\gamma) \end{array}\right\} \approx S \quad \Rightarrow$$



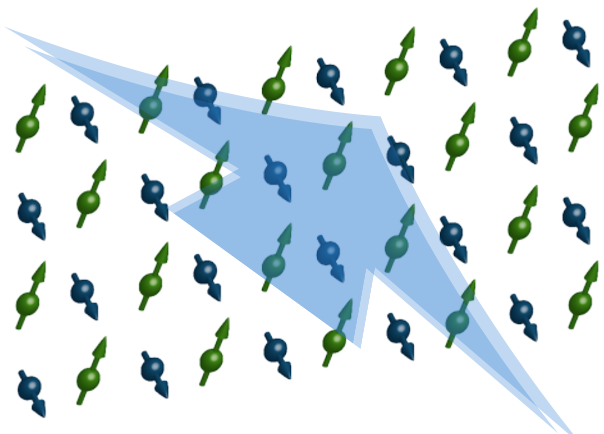
$$\gamma = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}$$

- #2: Criterion satisfied for ergodic states & local Hamiltonians

new Stein's lemma for ergodic states & local Gibbs states

$$\left.\begin{array}{l} \frac{1}{n} S^{\epsilon}_{\min}(\rho_n\,\|\,e^{-\beta H_n}) \\ \frac{1}{n} S^{\epsilon}_{\max}(\rho_n\,\|\,e^{-\beta H_n}) \end{array}\right\} \xrightarrow{n\to\infty} s(\rho) = \lim \frac{1}{n} S(\rho_n\,\|\,e^{-\beta H_n})$$

Putting everything together



ρ, ρ' ergodic
local Hamiltonian

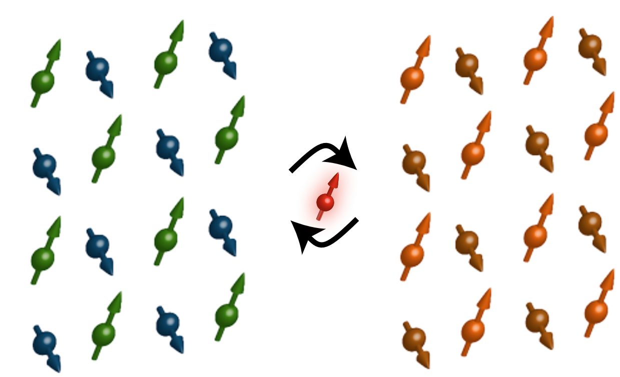
Step #2:

$$\left.\begin{array}{l} S_{\min}(\rho_n \,\|\, e^{-\beta H_n})/n \\ S_{\max}(\rho_n \,\|\, e^{-\beta H_n})/n \end{array}\right\} \to s(\rho)$$

$$s(\rho) = \lim_{n\to\infty} \frac{1}{n} S(\rho_n \,\|\, e^{-\beta H_n})$$

Step #1:

collapse of min- and max-relative entropies implies reversibility



& $s(\rho)$ is the emergent thermodynamic potential

Finite mixtures of ergodic states

Lemma:

$$S^{\epsilon}_{\min}(\sum p_k \rho^{(k)} \,\|\, \sigma) \sim \min_k S^{\epsilon}_{\min}(\rho^{(k)} \,\|\, \sigma)$$
$$S^{\epsilon}_{\max}(\sum p_k \rho^{(k)} \,\|\, \sigma) \sim \max_k S^{\epsilon}_{\max}(\rho^{(k)} \,\|\, \sigma)$$

Finite mixtures of ergodic states

Lemma:

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- A finitie mixture of ergodic states …
  - → … can be reversibly converted to/from the thermal state if all terms in the mixture have the same potential;
  - → … otherwise does not have a thermodynamic potential in the resource theory.